ON THE NONCOMMUTATIVE GEOMETRY OF APERIODIC SOLIDS

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Abstract. This mini-course, which will delivered at the INDAM Meeting *Noncommutative Geometry, Index Theory and Applications*, will offer a review of the achievements on a program of research that started in the Fall of 1979. The four lectures will concern the

(i) The Gap Labeling Theorem,
(ii) The Hull of aperiodic, repetitive, FLC tiling and the Anderson-Putnam Complex,
(iii) The transverse Geometry of tiling spaces,
(iv) Non FLC atomic distributions: a new approach to the Mechanics of real solids.

1. Plan of the Mini-Course

The four lectures will provide a summary of the following topics

1.1. **The Gap Labeling Theorem.** This part will give a description of the Hamiltonians associated with the electron dynamic in an aperiodic solid. Various examples will be provided, showing the existence of a Cantor spectrum. The integrated density if state (IDS) will be defined and his properties summarized. The C^*-algebra $\mathcal{A}$ generated by the Hamiltonian and its translated will be introduced, leading to the first notion of Hull. The Shubin formula is a crucial step in connecting the IDS to the K_0-group of $\mathcal{A}$. The gap labeling theorem will be established. Some general consequences will be discussed. A series of results for various examples of systems will conclude this part.

1.2. **The Hull and the Anderson-Putnam Complex.** Various examples of tilings will illustrate this part. Two constructions will be offered: the *cut-and-project* and the substitutions. Illustration by quasicrystal materials will be shown. The Hull will be redefined starting from the atomic configurations. Through the Voronoi construction it will be described by tilings. The Anderson-Putnam complex will be defined for substitution tilings, and the Hull can be identified as an inverse limit of such complexes. Examples in one and two dimension will be provided. Results concerning the calculation of the Čech cohomology of the Hull will be presented, using the *longitudinal cohomology*, the *pattern equivariant cohomology* or the *PV-cohomology*. A spectral sequence permits to compute the K-group starting from the Čech cohomology of the Hull. A description of the transversal and of its groupoid will be provided using the notion of *Michon tree* and of *Bratteli diagrams*, as a conclusion.

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1.3. **Transverse Geometry of the Hull.** The *Pearson-Palmer* spectral triple of a compact metric space will be described. It will be shown to provide informations about its Hausdorff dimension and Hausdorff measure, through the corresponding ζ-function. Applied to the transversal of a tiling space, it is related to the complexity of the tiling. The Pearson Laplacian will be described and few of its spectral properties provided. The extension of the spectral triple to the groupoid C^*-algebra will be discussed. This course will end with a discussion of the example of the Sierpinsky gasket, for which the efficiency of the spectral triple approach will be questioned, as many open problems remain mysterious.

1.4. **Non FLC Atomic Arrangements.** A quick description of Bulk Metallic Glasses (BMG) will begin the last lecture. The discussion of mechanical properties of solids from the microscopic scale will be addressed. Peridynamics will be mentionned, the description of composite materials also. It will be argued that only uniformly discrete set can provide a universal description of the atomic configurations. The results concerning the Hull will be summarized. The notion of Cantorization of the transversal will be discussed in the light of the structure of local patches. The *reconnection cohomology* will be introduced. In the concluding part a model describing the dynamics of such a solids, based on Thermodynamics, will be proposed and discussed.

2. Some History

The topic of this mini-course concerns a program of research that spanned a period of 33 years, since the Fall of 1979, when the author spent six months at the I.H.É.S, learning from Alain Connes what became later the *Noncommutative Geometry* [18]. During the eighties, the main source of concern was the description of the electronic motion in a nonperiodic solid, either in disordered systems [21, 42], such as the semiconductors at very low temperature [47], or with aperiodic order, such as the *quasicrystals* discovered in 1984 [46]. Several results where obtained concerning the electronic properties, in particular the existence of nowhere dense electronic spectrum [41, 3], at least in one dimension, or near the bottom of the spectrum in higher dimension [49]. The techniques developed then where using the K.A.M. theorem [4], the trace map method [50, 7, 9] or a semiclassical method [8, 11].

One of the major surprising result was the *Gap Labeling Theorem*, linking the labeling of gaps, using the *integrated density of states* (IDS) to the K_0-group of the C^*-algebra generated by the Hamiltonian and its translated [2, 6, 10]. This led to the definition of the *Hull*, a foliated space, and of its *canonical transversal*, from the electronic point of view in 1984 [6, 10]. This theorem was obtained in several steps, starting in 1981 with the abstract version and few examples, such as the Harper model or the case of almost periodic potentials. Between 1984 and 1993, several properties were obtained, such as the Shubin formula, and other examples where provided in particular in the case of substitution sequences a complete calculation of the set of gap label became possible [10].

Between 1983 and 1994, another surprising result was obtained, describing from first principle, with the full power of the tools developed in Noncommutative Geometry, the Integer Quantum Hall Effect [12]. This theory, mathematically rigorous, is still today the only example of a physical phenomenon, having been experimentally validated, that is explained, I am tempted to say *that requires*, using Noncommutative Geometry.

In the nineties, several breakthrough were achieved. Continuing with physical problem a qualitative theory of transport in quasicrystals could be developed in relation with experimental results obtained on quasicrystals (see a review in [14]). Various techniques, like the Guarneri bound [25] and the multifractal analysis [22, 27, 28] permitted this development. On the other hand, a connection between the study of aperiodic solids and the tiling theory was proposed by Johannes Kellendonk between 1995 and 1997 [31, 32]. This breakthrough led to the construction, by Anderson and Putnam in 1998 [1], of a family of CW-complexes with inverse limit giving a topological construction of the Hull [15]. In particular its Čech cohomology was computed in several examples of substitution tilings. The arrival of this new generation of mathematician, including John Hunton, Lorenzo Sadun, and Jean-Marc Gambaudo, and later Marcy Barge, helped making substantial progress in the computation of the Hull and of its cohomology [44]. Using a spectral sequence analogous to the Atiyah-Hirzebruch one for CW-complexes, the computation of the K-theory became possible from the Čech cohomology of the Hull [24, 45]. For about ten years, the attention of this small community was focussed on understanding the properties of tiling that are *aperiodic, repetitive* with *finite local complexity* (FLC). A seminal series of papers by Lagarias, between 1999 and 2003 [34, 35, 36, 37], permitted to interpret the Hull as a compactification of the set of ground state atomic configurations [13]. Several results by Lorenzo Sadun and his collaborators, provided a frame to understand the role of the cohomology in the description of a tiling [44].

During the last six years, a third generation of younger scientist preparing their Ph.D. Thesis, joined this community and has already produced a substantial amount of progress. The use of spectral triples to describe the Transverse Geometry of the Hull, by analogy with the work of Connes and Moscovici on the Transverse Geometry of foliated manifolds [19, 20], has already given several important results, connecting the complexity of tilings with the fractal dimension of the transversal. In 2007, John Pearson also defined an analog of the Laplace-Beltrami Operator on the transversal, in the case it is a Cantor set, such as in the case of FLC tilings [43]. Jean Savinien, Antoine Julien and Johannes Kellendonk gave several important results about the spectrum of this Pearson Laplacian, in various cases of tiling, providing a fine study of the algorithmic complexity of a tiling [30, 33]. In addition, some obstruction were discovered recently to construct a spectral triple based on the C^*-algebra of the tiling rather than on the space of continuous functions on the transversal [17]. This question has been illustrated by the case of the crossed product of a C^*-algebra by a $\mathbb{Z}$-action, and showed that the prospect of getting such a description for a tiling space is still farfetched today.

From 1996 until very recently, the theory developed mainly from the mathematical standpoint. For indeed, the main examples of aperiodic materials studied by physicists where semiconductors at very low temperature and quasicrystals. In both cases the focus of attention shifted after 1995. In the case of semiconductors, the spectacular experiments performed in the seventies and the eighties, including the Quantum Hall effect, were essentially understood, and not much progress could be expected further. In addition, the Moore's Law, concerning the increase of power of the electronic devices as a function of the time, started being questioned as the limit of the technology was reached. The community of Solid State physics investigated other types of materials, such as *graphene* which are as periodic as it can be. Therefore the program of research on aperiodic solids became marginal in this domain. On the other hand, the performances of quasicrystals appeared to be disappointing in the mid nineties, to the point that funding agencies abandoned this program. Hence, for the community of aperiodic solids, who wanted to pursue in this direction, there was no option but concentrating on the development of the mathematical tool.

However, since 2000, a new class of materials has slowly been attracting the attention of expert in material sciences, the *bulk metallic glasses* (BMG) [29, 26, 23]. The terms refers to a series of alloys based on copper and zirconium, having a structure close to the one of glasses, such as the ones obtained with silicates, but with amazing elasticity properties. Several theoreticians have proposed to see the atoms in such alloys as hard spheres with a radius depending upon the atomic species [39, 40, 38]. But this description is too hard to lead to any quantitive description of the properties of such solids. In the end of the mini-course, it will be argued that the use of the Hull and of its transverse geometry, may actually be effective in proposing a model liable to describe such properties. In addition, the flexibility of the formalism is such that it might also provide a modeling that applies to the Continuum Mechanics of all solids, including the ones that might experience cracks or fractures, and the composite materials. In this direction, a new theory has slowly developed during the last 12 years, under the name of *peridynamics*, proposed by S.A. Silling in 2000 [48]. The hope is that the present mathematical description will become relevant to people developing the peridynamics point of view.

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