Lectures on coarse index theory

Thomas Schick*
Georg-August-Universität Göttingen
Germany

Cortona, June 2012

Abstract

Coarse index theory has been introduced by John Roe. It provides a theory to use tools from C^*-algebras to get information about the geometry of non-compact manifolds via index theory of Dirac type operators. Through the passage to the universal covering one also gets important information about compact manifolds, and indeed, the Baum-Connes assembly map becomes part of this theory. In the lecture series, we discuss the basic constructions of the theory and three particular applications: to partitioned manifolds (as introduced by Roe), to enlargeable manifolds (as defined by Gromov and Lawson) and to "the passage from the positive scalar curvature exact sequence to analysis. Along the way, we discuss several new index theorems.

Many of the basics are due to John Roe and Nigel Higson, the new results are joint with Hanke, Kotschick, Roe, Hanke and Pape, and with Paolo Piazza.

1 Outline of the course

We first introduce the basic players: we start with a positive dimensional spin manifold X, assign to it the Roe algebras $C^*(X) \subset D^*(X)$. Using an direct and easy construction in C^*-algebras, one defines the Roe-index of the Dirac operator in $K_*(C^*(X))$. If the manifold has positive scalar curvature, this index vanishes. Instead, one gets a secondary invariant $\rho(g) \in K_*(D^*(X))$ which contains information about the positive scalar curvature metric (and can, e.g., distinguish between different components of the space of all such metrics). We will develop this as a recurring theme: a geometric reason for the vanishing of an index should indeed always give rise to a secondary invariant which gives new information about the space.

Given an isometric action by a discrete group Γ on X, we get algebras of invariants $C^*(X)^\Gamma \subset D^*(X)^\Gamma$ and a refined index/secondary invariant

*e-mail: schick@uni-math.gwdg.de
www: http://www.uni-math.gwdg.de/schick

in the K-theory of these algebras. There is an important map induced by "forgetting the equivariance".

We obtain such an isometric action in particular on covering spaces. This is an interesting way to assign non-compact manifolds with extra structure to compact manifolds, and via this route coarse geometry gives information about compact manifolds. Moreover, it is not hard to see that the Baum-Connes assembly map is just a special case of the general constructions of coarse index theory.

The K-theory of $C^*(X)$ and $D^*(X)$ (and there equivariant cousins) admits tools for computation. In particular it is functorial for rather general maps of spaces, and we will establish a Mayer-Vietoris sequence. We will also establish some vanishing results, e.g. for spaces of the form $X \times [0,\infty)$ (with product metric). Crucial here (and a common theme in the application of C^*-algebra techniques) is the implementation of geometric data —here the subspaces into which a space is decomposed— by suitable C^*-ideals.

We will exploit functoriality and the computation machine to explain and refine the obstruction to positive scalar curvature due to "enlargeability" (introduced by Gromov-Lawson) in terms of the standard index obtained via the Baum-Connes assembly map (the Rosenberg or Mishchenko index), with values in the K-theory of the group C^*-algebra (this is joint with Hanke, Kotschick, Roe).

We then obtain, as a first application, the partitioned manifold index and the partitioned manifold index theorem: Whenever X can be partitioned with a two-sided hypersurface M which is compact (or on which Γ acts cocompactly) then the Dirac operator on X defines a partitioned manifold index in $K_{*-1}(C^*(M)^\Gamma)$. The partitioned manifold index theorem (proved for $\Gamma = \{1\}$ by Roe and Higson, for general Γ and even dimension by Esfahani-Zadeh) states that this partitioned manifold index coincides with the one of M. As an application, we obtain that non-vanishing of the index of the hypersurface M is an obstruction to uniformly positive scalar curvature on X.

Next, we will discuss how one can use information like positive scalar curvature given only on part of the manifold. It turns out that C^*-algebraic methods also here apply very efficiently: by an appropriate choice of intermediate C^*-algebras (here, between $C^*(X)$ and the trivial one) one can exploit the additional information and sometimes get vanishing results. As an application we will present a refined obstructions to positive scalar curvature coming from submanifolds of codimension 2, in the spirit of Gromov-Lawson but applicable in more general situations and explaining the theory using C^*-index techniques. This is joint with Hanke and Pape.

Next, we will present a discussion of bordism invariance of the index of the Dirac operator in the language of coarse geometry. As discussed above: a basic feature is the implementation of the geometric situation (here: the manifold, its boundary, but also the relative theory of the manifold relative to its boundary) as ideals within one big algebra (D^*X). Indeed, this approach allows to get a secondary invariant of the bordism (some kind of relative fundamental class) which explains the vanishing of the index of the Dirac operator of a bounding manifold. Although this is certainly well known to the experts, such an approach to bordism invariance does not seem to be part of the available literature.

Finally, we will turn our attention to new index theorems (obtained jointly with Paolo Piazza). The methods allow to define in a canonical way a $C^*(\Gamma)$-index of the Dirac operator for a spin manifold X with boundary and with fundamental group Γ, provided the boundary has positive scalar curvature (and product structure).

We get a secondary Atiyah-Patodi-Singer index theorem for this coarse index: its "delocalized" part, i.e. the image in the K-theory of $D^*\Gamma$, coincides with the rho-invariant of the boundary.

Similarly, we obtain a secondary partitioned manifold index theorem: if X is a partitioned manifold as above with uniformly positive scalar curvature, then one has (as in the primary case) a partitioned manifold rho-invariant.

The index theorem say that this rho-invariant coincides with the rho-invariant of a partitioning hypersurface. We discuss the (surprisingly difficult) proofs of these theorems.

Finally, we give as an application a direct construction, using index theory, of a transformation from the positive-scalar-curvature exact sequence of Stolz to the long exact sequence of the pair $C^* \subset D^*$. This establishes a "map for psc to analysis", in analogy to Higson-Roe's construction of a map from "surgery to analysis". The construction presented here is even more directly using methods from index theory and C^*-algebras.