Master Thesis

The Exact Embedding Theorem

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Contents

1 Introduction

In the late seventies, J. Cuntz introduced a class of particularly interesting C^*-algebras, the so called Cuntz algebras $\mathcal{O}_n$. Approximately twenty years later, E. Kirchberg and N. C. Phillips published an article containing an embedding theorem for separable exact C^*-algebras into the second Cuntz algebra. Not only is this result fascinating in itself, it also entails numerous other important results. To these belong Kirchberg's $A \otimes \mathcal{O}_2$ theorem, giving a characterization of simple, separable, unital and nuclear C^*-algebras by means of an absorbing property, and the classification theorem for Kirchberg algebras, proved independently by Kirchberg and Phillips. The core of this thesis is Kirchberg's exact embedding theorem.

My principal aim was to make this exposition as self contained and comprehensible as possible. In view of the manifold material touched upon in order to prove the exact embedding theorem and the fact that some of the material here gathered is rather scattered over the literature, it will not come to surprise that this manuscript grew to the present size.

The contents in this thesis are mainly the basics of purely infinite C^*-algebras, including a rather detailed description of comparison theory, being necessary since I tried to describe the rudiments of non-simple purely infinite C^*-algebras, as well as facts needed about the real rank of C^*-algebras, discrete crossed products to the extent required for the exact embedding theorem and, of course, a proof of the embedding theorem itself, following closely the route described by M. Rørdam in [Rr1]. Results concerning ultrapower algebras and quasidiagonality have been banned to the appendix, yet complete proofs are given, apart from Voiculescu's theorem on the homotopy invariance of quasidiagonality.

My principle reference is the excellent book by G. Murphy, which actually helped me survive in operator theory for astonishingly long time, and naturally the very readable, even though concise, book by M. Rørdam. I have tried to give references to the original publications as often as possible; however, in the course of the proof of the exact embedding theorem I have adopted the habit of refering to M. Rørdam's book and to the original paper – at least where some similarity was detectable. Apart from that, I have tried to keep the number of references needed to a minimum.

I thank mainly Prof. M. Rørdam for giving me an interesting and topical field to work on and for continuous help and encouragement over the last year. I am also grateful to Prof. H. König for support on 'the german' side and for making a thesis in Denmark possible. Thanks further appertain to my cousin for continuous interest in anything new and many valuable discussions and questions.

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2 Preliminaries

In the sequel, A will always denote a C^*-algebra. We begin by recalling some definitions and fixing notation.

Denote by $\sigma(a)$, $\rho(a)$ the spectrum and spectral radius of an element $a \in A$, respectively, by $\mathcal{P}(A)$ the set of projections in A, by A_{sa} the set of self adjoint elements in A, and by A^+ the cone of positive elements in A. The C^*-algebra of $n \times n$ matrices over A will be denoted by $M_n(A)$ and their settheoretic union as $M_\infty(A) := \bigcup_{n\in\mathbb{N}} M_n(A)$, for a unital algebra we write 1_n for the unit in $M_n(A)$, and we abbreviate $M_n(\mathbb{C})$ to M_n; hence, using the identification $M_n \otimes A \cong M_n(A)$ we let $1_n \otimes a$ denote the matrix in $M_n(A)$ having diagonal entrys a and zeros elsewhere. Further, given another C^*-algebra B, $A \otimes B$ will always mean the minimal or spatial tensor product (see [M] for proofs concerning tensor products). The space of states on A will be denoted by $\mathcal{S}(A)$, and the subspace of pure states by $\mathcal{PS}(A)$. Given $a, b \in A$, we write $a \oplus b$ for the matrix in $M_2(A)$ with a and b on the diagonal and zeros elsewhere.

We call a map $f\colon A \to A$ between C^*-algebras a *-conjugation if there is an $a \in A$ such that $f(x) = a^*xa$ for every $x \in X$, and then a^*xa will be called the *-conjugate of x by a. We further use the notation $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

We assume the reader is familiar with the notions of tensor products of C^*-algebras, nuclearity and exactness as well as with completely positive and completely bounded maps.

The next two sections contain general results which will be used in the sequel.

2.1 On Hilbert spaces

Remember that a closed subspace $\mathcal{K}$ of a Hilbert-space $\mathcal{H}$ is called invariant under a bounded operator a on $\mathcal{H}$ if $a(\mathcal{K}) \subseteq \mathcal{K}$, and reducing for a if both $\mathcal{K}$ and $\mathcal{K}^\perp$ are invariant under a.

Observation 2.1. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K} \subseteq \mathcal{H}$ a closed subspace. Let $a \in \mathcal{B}(\mathcal{H})$ and denote by p the orthogonal projection onto $\mathcal{K}$.

(i) $\mathcal{K}$ is invariant for a iff $pap = ap$.

(ii) $\mathcal{K}$ is invariant for a iff $\mathcal{K}^\perp$ is invariant for a^*.

(iii) $\mathcal{K}$ is reducing for a iff p commutes with a.

Proof. (i) If $\mathcal{K}$ is invariant for a, i.e. $a\mathcal{K} \subseteq \mathcal{K}$, then $pa\xi = a\xi$ for all $\xi \in \mathcal{K}$, therefore we get for all $\xi \in \mathcal{H}:\ pap\xi = ap\xi$.

Conversely, $pap = ap$ implies $pa|_\mathcal{K} = a|_\mathcal{K}$, hence $a\xi \in \mathcal{K}$ for all $\xi \in \mathcal{K}$.

(ii) Assume $\mathcal{K}$ is invariant for a, i.e. $pap = ap$ by (i). Then

$$(1-p)a^*(1-p) = a^* - pa^* - a^*p + pa^*p = a^* - pa^* - a^*p + a^*p = a^*(1-p),$$

and consequently $\mathcal{K}^\perp = \mathrm{Im}(1-p)$ is invariant for a^* by 1). As $\mathcal{K}$ is closed, we may apply this to see that if $\mathcal{K}^\perp$ is invariant for a^*, then $\mathcal{K}^{\perp^\perp} = \mathcal{K}$ is invariant for a.
(iii) $\Rightarrow$: Let $\mathcal{K}$ be reducing for a, i.e. $\mathcal{K}$ and $\mathcal{K}^\perp$ are a-invariant. Then $\mathcal{K}$ is a- and a^*-invariant, implying by (i):

$$ap = pap = (pa^*p)^* = (a^*p)^* = pa.$$

$\Leftarrow$: Let $pa = ap$, then $pap = ap$ and $(1-p)a(1-p) = a(1-p)$ and by (i) $\mathcal{K}$ is reducing for a. $\square$

Proposition 2.2. *Let $a_1, \ldots, a_n \in A$. Then the following equalities hold:*

(i) $||\operatorname{diag}(a_1, \ldots, a_n)|| = \max\{||a_1||, \ldots, ||a_n||\}$

(ii) $|| \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} || = ||\operatorname{diag}(a_1, a_2)||$.

Proof. To see that (i) holds observe that

$$\varphi\colon \bigoplus_{k=1}^{n} A \to M_n(A);\ (x_1, \ldots, x_n) \mapsto \operatorname{diag}(x_1, \ldots, x_n)$$

is an injective *- homomorphism and hence isometric (where we use the standard sup-norm on the sum over A).
Concerning (ii) we have

$$\begin{aligned} \left|\left| \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \right|\right|^2 &= \left|\left| \begin{pmatrix} 0 & a_2^* \\ a_1^* & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \right|\right| \\ &\overset{(i)}{=} \max\{||a_1^* a_1||, ||a_2^* a_2||\} \\ &= \max\{||a_1||, ||a_2||\}^2 \end{aligned}$$

as claimed. $\square$

Remark 2.3. Let $a \in A$ and $p \in \mathcal{P}(A)$. Set

$$a_1 := pap,\ a_2 := pa(1-p)\ a_3 := (1-p)ap,\ a_4 := (1-p)a(1-p),$$

which entails

$$a_1 + a_2 + a_3 + a_4 = p(ap + a(1-p)) + (1-p)(ap + a(1-p)) = a.$$

As all of the subalgebras pAp, $(1-p)A(1-p)$, $(1-p)Ap$, and $pA(1-p)$ have empty intersection, we have shown that A is isomorphic to the inner direct some of them (as a vector space).

Interpreting this in Hilbert space terms, that is, representing A faithfully on some Hilbert space $\mathcal{H}$ which may be written as direct sum of $\mathrm{Im}(p)$ and $\mathrm{Im}(p)^\perp$, we have thus decomposed a into parts $a_1\colon \mathrm{Im}(p) \to \mathrm{Im}(p)$, $a_2\colon \mathrm{Im}(p) \to \mathrm{Im}(p)^\perp$, $a_3\colon \mathrm{Im}(p)^\perp \to \mathrm{Im}(p)$ and $a_4\colon \mathrm{Im}(p)^\perp \to \mathrm{Im}(p)^\perp$ and a may now be viewed as a matrix $\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ which again gives a bounded linear mapping under the identification of $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathrm{Im}(p) \oplus \mathrm{Im}(p)^\perp)$ with 2×2-matrices having entries as above.
The above decomposition of elements in C^*-algebras with respect to some projection p is a standard procedure the which will be used in the sequel without further comment. As another demonstration of this technique we have:

Remark 2.4. If $p \in \mathcal{P}(A)$ we have for all $a \in A$:

$$||\big[a, p\big]|| = ||ap - pa|| = \max\{||(1-p)ap||, ||pa(1-p)||\}.$$

This follows, since applying the above decomposition for a to $\big[a, p\big]$, we obtain a 2×2 matrix with only off-diagonal entrys and may therefore use Proposition 2.2 (ii).

Let again $a \in A$ and $p \in \mathcal{P}(A)$. Then if $a \in pAp$, a will be called an element in the corner pAp. In fact, a will correspond to a matrix having only one entry in the upper left corner when we use the decomposition as above. We will also refer to elements in $(1-p)Ap$ and $pA(1-p)$ as elements in the corner $(1-p)Ap$ and $pA(1-p)$, respectively. Note also that p is a unit for pAp.

2.2 Miscellaneous

Lemma 2.5. *For projections $p_1, \dots, p_n$ in a C^*-algebra A the following conditions are equivalent:*

(i) *the projections $p_1, \dots, p_n$ are mutually orthogonal*

(ii) $\sum_{k=1}^n p_k$ *is a projection*

(iii) $\sum_{k=1}^n p_k \leq 1$

Proof. Obviously every condition implies the one below it. It remains to show that (iii) implies (i). Let $1 \leq i, j \leq n$ and $i \neq j$. As $p_i + p_j \leq 1$, we have $p_i(p_i + p_j)p_i \leq p_i$, giving $p_i p_j p_i \leq 0$. We deduce

$$||p_i p_j||^2 = ||p_i p_j p_i|| = 0\,,$$

as desired. □

Lemma 2.6. *Let I be an algebraic ideal in A. Then for any projection $p \in \bar{I}$ we have $p \in I$. In addition, there is an element $a \in I$ such that p lies in the two sided ideal AaA generated by a.*

Proof. Choose some $a \in I$ such that $||p - a|| < 1$. Then $pap \in I$ and $||p - pap|| < 1$, implying invertibility of pap in pAp. Choose an inverse $b \in pAp$ of pap in pAp. Then

$$p = bpap \in AaA.$$

□

Lemma 2.7. *Let A, B be C^*-algebras. For every nonzero projection $p \in \mathcal{P}(B)$ there is an embedding*

$$\iota_p \colon A \to A \otimes B,\ a \mapsto a \otimes p\,.$$

Proof. ι_p is linear as $\otimes$ is bilinear. Let $a, b \in A$. Then we have that

$$\iota_p(ab) = ab \otimes p = (a \otimes p)(b \otimes p) = \iota_p(a)\iota_p(b)$$

and

$$\iota_p(a^*) = a^* \otimes p = (a \otimes p)^* = \iota_p(a)^*,$$

implying that ι_p is a *-homomorphism. It remains to show injectivity. Assume that $(a - b) \otimes p = \iota_p(a) = 0$, then

$$0 = ||(a - b) \otimes p|| = ||a - b||\,||p|| = ||a - b||$$

implies $a = b$. □

Lemma 2.8. *Let A, B be C^*-algebras and $(a, b) \in A \oplus B$, $f \in C\big(\sigma((a, b))\big)$ with $f(0) = 0$. Then $f((a, b)) = (f(a), f(b))$.*

Proof. We may approximate f uniformly by a sequence $(p_n)_n$ of polynomials without constant term, and for these the statement holds by definition of the operations in the sum $A \oplus B$, hence it holds for the limit f . □

Proposition 2.9. *Let A be unital, $p \in \mathcal{P}(A)$ and $a \in A$ such that $ap = a$ and $||a^*a - p|| < 1$. Then, evaluating the functional calculus in pAp, $v := a(a^*a)^{-1/2}$ gives an element in A with $v^*v = p$. Moreover:*

$$||v - a|| \leq 1 - (1 - ||a^*a - p||)^{1/2} \leq ||a^*a - p||\,.$$

Proof. We have $ap = a$ and $pa^* = a^*$, hence we get $a^*a = pa^*ap \in pAp$. Now a^*a is positive in A and $\sigma_A(a^*a) \cup \{0\} = \sigma_{pAp}(a^*a) \cup \{0\}$ by [M, Theorem 1.2.8], so a^*a is positive in pAp. Therefore $(a^*a)^{1/2}$ exists and $||a^*a - p|| < 1$ implies invertibility of a^*a in pAp, whereby $(a^*a)^{-1/2}$ exists in pAp. Then setting $v := a(a^*a)^{-1/2}$ we have $v^*v = (a^*a)^{-1/2*}a^*a(a^*a)^{-1/2} = p$ and therefore v is a partial isometry in A.
To show the inequality. First observe

$$\begin{aligned}(v - a)^*(v - a) =& v^*v - v^*a - a^*v + a^*a\\ =& p - (a^*a)^{-1/2}a^*a - a^*a(a^*a)^{-1/2} + a^*a\\ =& p - p(a^*a)^{1/2} - (a^*a)^{1/2}p + a^*a\\ =& (p - (a^*a)^{1/2})^2\,.\end{aligned}$$

Set $f(t) := t^{1/2} - 1$ and observe that $(a^*a)^{1/2} - p = f(a^*a)$, evaluating the functional calculus in pAp. Define $\delta := ||a^*a - p||$ and note further that $\sigma_{pAp}(a^*a) \subseteq [1-\delta, 1+\delta]$. We deduce:

$$\begin{aligned} ||v - a|| = ||(a^*a)^{1/2} - p|| =& ||f|_{\sigma(a^*a)}||_\infty \leq \sup\{|t^{1/2} - 1| \mid t \in [1-\delta, 1+\delta]\} \\ =& 1 - (1-\delta)^{1/2} = 1 - (1 - ||a^*a - p||)^{1/2}, \end{aligned}$$

and hence the first inequality holds. The second follows as $(1-\delta)^{1/2} \geq 1-\delta$, implying:

$$1 - (1-\delta)^{1/2} \leq \delta\,.$$

□

We denote by $\mathbb{T}$ the unit sphere in the complex plane.

Proposition 2.10. *Let A be a unital C^*-algebra containing a unitary u. Then there is a unital *-homomorphism $\varphi : C(\mathbb{T}) \to A$ which is injective if and only if the spectrum of u is the full unit circle.*

Proof. If K denotes the spectrum of u, then restriction to K is a unital *-homomorphism from $C(\mathbb{T})$ onto $C(K)$, which is clearly injective if and only if $K = \mathbb{T}$. Now the continuous functional calculus provides a unital *-homomorphism $\psi : K \to C^*(u)$, where $C^*(u)$ denotes the sub-C^*-algebra generated by u. The map sending $f \in C(\mathbb{T})$ to $\psi(f|_K) =: \varphi(f)$ therefore has the desired properties. □

Lemma 2.11. *Let A be a unital C^*-algebra containing a non-unitary isometry s. Then $\sigma(s + s^*) = [-2, 2]$.*

Proof. Assume without loss of generality, that $A \subseteq \mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Let $e_0 \in \mathrm{Im}(1 - ss^*)$ be a unit vector. Set $e_n := s^n e_0$. Then $s^* e_n = e_{n-1}$, as s is an isometry, and writing e_0 as $\xi - ss^*\xi$ for some $\xi \in \mathcal{H}$, we see that $s^* e_0 = s^*\xi - s^*ss^*\xi = 0$. It follows

$$\langle e_n \mid e_m \rangle = \langle s^n e_0 \mid s^m e_0 \rangle = \delta_{n,m}\,.$$

For every $\lambda \in \mathbb{T}$, define

$$\xi_{N,\lambda} := \frac{1}{N^{1/2}} \sum_{j=1}^{N} \lambda^j e_j\,,$$

and calculate

$$||\xi_{N,\lambda}||^2 = \frac{1}{N} \sum_{j=1}^{N} |\lambda^j|^2 = 1\,.$$

We have

$$\begin{aligned}
&N^{1/2}\big((s+s^*)\xi_{N,\lambda} - 2\operatorname{Re}(\lambda)\xi_{N,\lambda}\big)\\
=&\sum_{j=1}^{N}\lambda^j s e_j + \sum_{j=1}^{N}\lambda^j s^* e_j - 2\operatorname{Re}(\lambda)\sum_{j=1}^{N}\lambda^j e_j\\
=&\sum_{j=2}^{N+1}\lambda^{j-1}e_j + \sum_{j=0}^{N-1}\lambda^{j+1}e_j - \sum_{j=1}^{N}\lambda^{j+1}e_j - \sum_{j=1}^{N}\lambda^{j-1}e_j\\
=&\lambda^N e_{N+1} - e_1 + \lambda e_0 - \lambda^{N+1}e_N\,.
\end{aligned}$$

This shows that

$$\eta_{n,\lambda} := (s+s^*)\xi_{N,\lambda} - 2\operatorname{Re}(\lambda)\xi_{N,\lambda} \to 0$$

for $N \to \infty$. Assume now that $a := (s + s^* - 2\operatorname{Re}(\lambda))^{-1}$ exists, and deduce

$$||\xi_{n,\lambda}|| = ||a\eta_{n,\lambda}|| \leq ||a||\ ||\eta_{n,\lambda}|| \to 0$$

for $n \to \infty$, and therefore a is not bounded, hence $2\operatorname{Re}(\lambda) \in \sigma(s+s^*)$. □

Proposition 2.12. *Let A be a unital C^*-algebra containing a non-unitary isometry s. Then $C(\mathbb{T})$ embeds unitally into A.*

Proof. As $a := \frac{1}{2}(s+s^*)$ has spectrum $[-1,1]$ by Lemma 2.11, the element $\exp(i\pi a)$ is a unitary with spectrum exactly $\mathbb{T}$. By Proposition 2.10 we have an embedding of $C(\mathbb{T})$ into A. □

3 Comparison theory and finiteness conditions

3.1 Comparison theory

In the following we introduce a couple of relations between positive elements in C^*-algebras and deduce some basic properties of these. They will be used to define the notions of finite, infinite and properly infinite positive elements in the next section. Afterwards we devote a section to relate these to the classical comparison theory for projections introduced by Murray and von Neumann.
We will first remind the reader of the classical order relation $\leq$ on C^*-algebras and related definitions. Of course, this has been used in section one, but we recall these results here to stress the connections with more advanced comparison theory - the context in which this subject naturally belongs.
An element $a \in A$ is called positive ($a \geq 0$), if $\sigma(a) \subseteq \mathbb{R}^+$, and we will denote by A^+ the cone of positive elements in A; a classical theorem then says that $A^+ = \{a^*a \mid a \in A\}$ (see [M, Theorem 2.2.5] 2.2 for a proof). For every positive element a in A we may use the Gel'fand theorem ([M, Theorem 2.1.13]) to construct the square root $a^{1/2}$ of a (note also that by construction $a^{1/2}$ commutes with a). Therefore, if a is contained in a C^*-subalgebra B of A, then $a^{1/2} \in B$ also, and as $a = (a^{1/2})^* a^{1/2}$ we deduce that a is positive in B.
In addition, for every element a in A, we have $a^*a \geq 0$, and may set $|a| := (a^*a)^{1/2}$.
We quote the following standard theorem, proved, for example, in [M, Theorem 2.3.4], for easy reference:

Theorem 3.1 (Polar decomposition). *Let a be a bounded operator on a Hilbert space $\mathcal{H}$. Then there is a unique partial isometry $v \in B(\mathcal{H})$ such that:*

$$a = v|a|, \qquad \mathrm{Ker}(v) = \mathrm{Ker}(a),$$

further $v^*a = |a|$ holds.

In the case where the operator a is invertible, the theorem has an easy proof and v is a unitary: First observe that by functional calculus the element $|a|$ is invertible too, and the same holds for $v := a|a|^{-1}$. Then obviously $a = v|a|$ holds and

$$v^*v = |a|^{-1}a^*a|a|^{-1} = 1\,,$$

implying that v is unitary (right and left inverse coincide, if they exist). By uniqueness, this is the partial isometry in Theorem 3.1. Of course, we could have also used Proposition 2.9.

We will need the notion of approximate unit and hereditary C^*-subalgebra and therefore give a short account of their structure.
For any C^*-algebra A there is an increasing net u_λ of positive elements in the closed unit ball such that $\lim_\lambda(au_\lambda - a) = 0$ for all $a \in A$ (in fact, the set of positive elements contained in the closed unit ball itself will do). A net with these properties will be called an increasing approximate unit; if the assumption that the net is increasing is dropped, we will simply refer to it as an approximate unit.
A hereditary C^*-subalgebra H in A is C^*-subalgebra with the property that for all $a, b \in A^+$ such that $a \leq b$ and $b \in H$ we have $a \in H$. There is a close relation between closed left ideals L in A and hereditary C^*-subalgebras, namely that every hererditary C^*-subalgebra H corresponds to exactly one such L and may be reobtained as $L \cap L^*$. Furthermore, if we denote the smallest hereditary C^*-subalgebra containing a subset X of A by $\mathrm{Her}(X)$, then $\mathrm{Her}(a) = \overline{aAa}$ for every positive element $a \in A$; the converse, that is, all hereditary sub-C^*-algebras of A are of the form $\overline{aAa}$, holds for separable C^*-algebras. A proof for the following characterization, and likewise for the rest of this short introduction, may be found in [M, chapter 3].

Theorem 3.2. *Let H be a C^*-subalgebra of A. Then H is hereditary if and only if $bab' \in H$ for all $a \in A$ and $b, b' \in H$.*

The next lemma will be used to justify the definition following it:

Lemma 3.3 (cf. [Ped3, Lemma 3.1]). *Let $\mathcal{H}$ be a Hilbert space and A a C^*-subalgebra in $B(\mathcal{H})$. If $a \in A_{sa}$ and va remains in A for some $v \in B(\mathcal{H})$, then for every $f \in C(\sigma(a))$ with $f(0) = 0$ the element $vf(a)$ is an element of A.*

Proof. Use the Weierstraß theorem ([Ped1, Theorem 4.3.3] to find a sequence of polynomials p_n such that p_n converges uniformly to f on $\sigma(a)$. As

$$|p_n(0)| = |p_n(0) - f(0)| \to 0$$

for $n \to \infty$, we may use the sequence $\tilde{p}_n := p_n - p_n(0)$ instead to obtain a sequence of polynomials which still tends to f uniformly on $\sigma(a)$ and has no constant term; henceforth we may choose a sequence q_n of polynomials such that $\tilde{p}_n(t) = tq_n(t)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Then:

$$vf(a) = v \lim_{n\to\infty} p_n(a) = \lim_{n\to\infty} vaq_n(a) \in A\,.$$

□

Proposition 3.4 (cf. [Ped3, 5.2]). *We define an equivalence relation on A^+ by setting $a \sim b$ for two positive elements a and b if there is an element $x \in A$ such that $a = x^*x$ and $b = xx^*$. For positive elements $a, b \in M_\infty(A)$ we also write $a \sim b$ if there is a matrix x in some $M_{n,m}(A)$ with coefficients in A such that $a = x^*x$ and $b = xx^*$.*

Proof. Reflexivity follows as $a = a^{1/2}a^{1/2}$ and $a^{1/2} \in A_{sa}$, symmetry is obvious. We proceed to show transitivity, following the lines of [Ped3, Theorem 3.5].

Without loss of generality we may assume $A \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ ([M, Theorem 3.4.1]). Let $a, b, c \in A_+$ with $a \sim b \sim c$ and find $x, y \in A$ such that $a = xx^*$, $b = x^*x = y^*y$, $c = yy^*$ (and hence $|x| = (x^*x)^{1/2} = (y^*y)^{1/2} = |y|$). By Theorem 3.1 we may choose partial isometries $u, v \in B(\mathcal{H})$ such that:

$$x = u|x|, \qquad\qquad y = v|y|.$$

Then, first of all:

$$|x|v^*v|x| = (v|y|)^*v|y| = y^*y = x^*x = |x|^2. \tag{3.1}$$

Now set $z := vx^*$. Calculate

$$z^*z = u|x|v^*v|x|u^* \stackrel{3.1}{=} u|x|^2u^* = xx^* = a$$

and

$$zz^* = v|x|u^*u|x|v^* = v|x|^2v^* = v|y|^2v^* = yy^* = c\,,$$

showing that it now suffices to show $z \in A$ to get transitivity. As $|x|^{1/2} = |y|^{1/2}$ we may choose $w \in A$ (by Theorem 3.1) with

$$|x|^{1/2} = w||x|^{1/2}| = w|x|^{1/2} = w|y|^{1/2}\,,$$

and deduce:

$$z^* = u|x|v^* = u|x|^{1/2}w|y|^{1/4}|y|^{1/4}v^*.$$

Now all terms in this factorization are in A by 3.3, hence so is z. □

Remark 3.5. Note that $a, b \in A^+$ implies $a \oplus b \in A^+$.

Proposition 3.6. *The following properties hold for all $a, b, a', b' \in A^+$:*

(i) $a \oplus 0 \sim a$

(ii) $a \oplus b \sim b \oplus a$

(iii) *if $a \sim b$ and $a' \sim b'$, then $a \oplus b \sim a' \oplus b'$,*

(iv) *if $a \perp b$, then $a \oplus b \sim a + b$.*

Proof. Setting $v := \begin{pmatrix} a^{1/2} & 0 \end{pmatrix}$, we get:

$$v^*v = a \oplus 0\,, \qquad\qquad vv^* = a.$$

Hence (i) holds; (ii) follows from

$$\begin{pmatrix} 0 & a^{1/2} \\ b^{1/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & a^{1/2} \\ b^{1/2} & 0 \end{pmatrix}^* = a \oplus b$$

and

$$\begin{pmatrix} 0 & a^{1/2} \\ b^{1/2} & 0 \end{pmatrix}^* \begin{pmatrix} 0 & a^{1/2} \\ b^{1/2} & 0 \end{pmatrix} = b \oplus a.$$

To prove (iii), choose $x, y \in A$ such that $a = x^*x$, $a' = xx^*$, $b = y^*y$ and $b' = yy^*$. Then it follows that

$$\begin{pmatrix} 0 & x^* \\ y^* & 0 \end{pmatrix} \begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix} = a \oplus b, \qquad \begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix} \begin{pmatrix} 0 & x^* \\ y^* & 0 \end{pmatrix} = b' \oplus a'$$

and therefore (iii) holds by (ii).
To show (iv), observe first that if $a \perp b$, then for any polynomial p with $p(0) = 0$ we have $p(a) \perp p(b)$. Approximating the square root function by such polynomials (cf. the argument in Lemma 3.3), we see that $a^{1/2} \perp b^{1/2}$ if $a \perp b$. Consequently

$$\begin{pmatrix} a^{1/2} \\ b^{1/2} \end{pmatrix} \begin{pmatrix} a^{1/2} & b^{1/2} \end{pmatrix} = a \oplus b \qquad \begin{pmatrix} a^{1/2} & b^{1/2} \end{pmatrix} \begin{pmatrix} a^{1/2} \\ b^{1/2} \end{pmatrix} = a + b$$

as desired. □

Proposition/Definition 3.7. *For all $a, b \in A^+$ we set $a \precsim b$ if there is a sequence $(x_n)_n \in A^{\mathbb{N}}$ such that $x_n^* b x_n \to a$ as $n \to \infty$, thus defining an order relation on A^+.*

Proof. The relation $\precsim$ is reflexive, as, denoting the inclusion of $\sigma(a)$ into $\mathbb{C}$ by ι

$$a^{1/n} a a^{1/n} = (\iota^{1/n}\, \iota\, \iota^{1/n})(a) \to_{n \to \infty} \iota(a) = a,$$

using that $\sigma(a)$ compact.
We show that $\precsim$ is also transitive. Let $a, b, c \in A^+$ such that $a \precsim b \precsim c$. Choose sequences $(x_n)_n, (y_n)_n$ such that $x_n^* b x_n \to a$ and $y_n^* c y_n \to b$. For every $N \in \mathbb{N}$ we may then choose $m_N, n_N \in \mathbb{N}$ such that:

$$||x_{m_N}^* b x_{m_N} - a|| \leq \frac{1}{2N}, \qquad ||y_{n_N}^* c y_{n_N} - b|| \leq \frac{1}{2||x_{m_N}||^2 N}.$$

Setting $z_N := x_{m_N} y_{n_N}$ we obtain

$$||z_N^* c z_N - a|| \leq ||x_{m_N}||^2 ||y_{n_N}^* c y_{n_N} - b|| + ||x_{m_N}^* b x_{m_N} - a|| \leq \frac{1}{N},$$

and therefore $z_N^* c z_N \to_{N \to \infty} a$, implying $a \precsim c$. □

The next proposition shows that $\precsim$ is weaker than $\leq$:

Proposition 3.8 (cf. [Rr2, Lemma 2.3]). *Let $a, b \in A^+$. Then $a \leq b$ implies $a \precsim b$.*

Proof. For every $\delta > 0$ let

$$g_\delta \colon \mathbb{R}^+ \to \mathbb{R}^+;\ t \mapsto \min\{t^{-1}, \delta^{-1}\}.$$

Define $x_\delta := a^{1/2}(g_\delta(b))^{1/2}$ and $y_\delta := a^{1/2}(1 - g_\delta(b)\, b\,)^{1/2}$. Then we have:

$$y_\delta y_\delta^* = a^{1/2}(1 - g_\delta(b)\, b\,) a^{1/2} = a - a^{1/2} g_\delta(b) b a^{1/2} = a - x_\delta b x_\delta^*,$$

and, using [M, Theorem 2.2.5] at $(*)$:

$$y_\delta^* y_\delta = (1-g_\delta(b)b)^{1/2} a (1-g_\delta(b)b)^{1/2} \overset{(*)}{\leq} (1-g_\delta(b)b)^{1/2} b (1-g_\delta(b)b)^{1/2}$$
$$= (1-g_\delta(b)b)b\,.$$

As for all $t \in [0,\delta]$ we have $g_\delta(t) = \frac{1}{\delta}$ and $g_\delta(t)t = 1$ for all $t \geq \delta$, it follows that

$$||(1-g_\delta \cdot \mathrm{id})|_{\mathbb{R}^+}||_\infty = 1$$

and

$$||(1-g_\delta(b)b)b|| = ||\big((1-g_\delta \cdot \mathrm{id}) \cdot \mathrm{id}\big)\big|_{[0,\delta]}||_\infty \leq \delta.$$

Therefore:

$$\lim_{\delta \to 0} ||a - x_\delta^* b x_\delta|| = \lim_{\delta \to 0} ||y_\delta y_\delta^*|| = \lim_{\delta \to 0} ||(1-g_\delta(b)b)b|| = 0\,.$$

□

Lemma 3.9. *Let $a \in A^+$, and let $\varphi\colon \sigma(a) \to \mathbb{R}^+$ be a continuous function with $\varphi(0) = 0$. Then $\varphi(a) \precsim a$.*

Proof. Use the Weierstraß theorem [Ped1, Theorem 4.3.3] to find a sequence of polynomials without constant term $p_n(t)$ that converges uniformly to φ on $\sigma(a)$ (cf. Lemma 3.3), and find a sequence $\tilde{p}_n$ of polynomials with $p_n(t) = t\tilde{p}_n(t)$. Then $\tilde{p}_n(a)^{1/2}$ commutes with a and therefore

$$\tilde{p}_n(a)^{1/2} a \tilde{p}_n(a)^{1/2} = p_n(a) \to \varphi(a)$$

as $n \to \infty$, implying $\varphi(a) \precsim a$. □

Proposition/Definition 3.10. *For all $\varepsilon \geq 0$ set $\varphi_\varepsilon := \max\{0, \mathrm{id} - \varepsilon\}$ and define $(a-\varepsilon)_+ := \varphi_\varepsilon(a)$ for all $a \in A_{sa}$. Then the following properties hold for all $a \in A_{sa}$:*

(i) $0 \leq (a-\varepsilon)_+ \leq a$ *if* $a \geq 0$,

(ii) $((a-\varepsilon_1)_+ - \varepsilon_2)_+ = (a-(\varepsilon_1+\varepsilon_2))_+$ *for all* $\varepsilon_1, \varepsilon_2 \geq 0$,

(iii) $(a-\varepsilon)_+ \to a_+$ *as* $\varepsilon \to 0$,

(iv) $a_+ \perp (-a)_+$ *and* $a = a_+ - (-a)_+$, $|a| = a_+ + (-a)_+$,

(v) $a \geq 0$ *if and only if* $a = a_+$,

(vi) *If A is unital and $0 \notin \sigma(a)$, then there is a projection $p \in A$ with $pa = ap = a_+$,*

(vii) *If B is another C^*-algebra and $(a,b) \in (A \oplus B)_{sa}$, then:*

$$((a,b)-\varepsilon)_+ = ((a-\varepsilon)_+, (b-\varepsilon)_+)\,.$$

Proof. (i) follows directly from the spectral mapping theorem ([M, Theorem 2.1.14]), (ii) is easily seen as

$$((a-\varepsilon_1)_+ - \varepsilon_2)_+ = \varphi_{\varepsilon_2}(\varphi_{\varepsilon_1}(a)) = \varphi_{\varepsilon_1+\varepsilon_2}(a),$$

and (iii) follows from

$$||(a-\varepsilon)_+ - a_+|| = ||(\varphi_\varepsilon - \varphi_0)(a)|| = ||\varphi_\varepsilon - \varphi_0||_\infty \to_{\varepsilon\to 0} 0\,.$$

(iv) is a consequence of $\varphi_0(\cdot) \perp \varphi_0(-\cdot) = 0$ and $\varphi_0(\cdot) - \varphi_0(-\cdot) = \mathrm{id}$; (v) follows as $a \geq 0$ if and only if $\varphi_0|_{\sigma(a)}$ is the inclusion of the spectrum of a. To prove (vi), take f as the function being constant 1 on $\sigma(a) \cap \mathbb{R}^+$ and constant 0 on $\sigma(a) \cap -\mathbb{R}^+$, and set $p := f(a)$.
(vii) is a special case of Lemma 2.8. □

In view of Lemma 3.10 one defines $a_- := \varphi_0(-a)$ and calls a_+, a_- the positive and negative part, respectively, of the element a.

Lemma 3.11 (cf. [Rr2, Lemma 2.2]). *Let $\varepsilon > 0$ and $a, b \in A^+$ such that $||a-b|| < \varepsilon$. Then $(a-\varepsilon)_+ \leq xbx$ for some positive element x in A.*

Proof. Set $||a-b|| := \delta$. Then $\delta - (a-b) \geq 0$, giving $b \geq a - \delta$. As $\varphi_\varepsilon(t) = 0$ for all $t \leq \varepsilon$ and $\varepsilon - \delta \leq t - \delta$ for all $t \geq \varepsilon$ we have $(\varepsilon-\delta)\varphi_\varepsilon \leq (id - \delta)\varphi_\varepsilon$. We deduce:

$$(\varepsilon-\delta)(a-\varepsilon)_+ \leq (a-\delta)(a-\varepsilon)_+ = (a-\varepsilon)_+^{1/2}(a-\delta)(a-\varepsilon)_+^{1/2} \leq (a-\varepsilon)_+^{1/2} b (a-\varepsilon)_+^{1/2}.$$

Now set $x := (\varepsilon-\delta)^{-1/2}(a-\varepsilon)_+^{1/2}$ to complete the proof. □

Observation 3.12. For every $\varepsilon > 0$ one may define the function $h_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ to be used in the following lemma as being constant zero on $[-\infty, \varepsilon]$, constant one on $[2\varepsilon, \infty]$ and linear on $[\varepsilon, 2\varepsilon]$. Observe that $h_{\varepsilon_1} h_{\varepsilon_2} = h_{\varepsilon_1}$ as long as $\varepsilon_1 \geq 2\varepsilon_2$. Also for all $a \in A_{sa}$ there is some $x \in A^+$ such that $h_{\varepsilon_1}(a) = x(a-\varepsilon_2)_+ = x^{1/2}(a-\varepsilon_2)_+ x^{1/2}$ as long as $\varepsilon_1 \leq \varepsilon_2$, and $(a-\varepsilon_1)_+ = y h_{\varepsilon_2}(a) = y^{1/2} h_{\varepsilon_2}(a) y^{1/2}$ for some $y \in A^+$.

We will further need the following Proposition:

Proposition 3.13 (cf. [Ped2, Proposition 1.4.5]). *Let a and b be elements in a C^*-algebra A such that $a \geq 0$ and $a^*a \leq b$. If $0 < \alpha < \frac{1}{2}$, then there is an element x in A with $||x|| \leq ||b^{1/2-\alpha}||$ such that $a = xb^\alpha$.*

The following lemma is one of the key results in this section:

Lemma 3.14 (cf. [KiRr2, Lemma 2.5]). *Let $a, b \in A^+$. If $\varepsilon > 0$ such that $\delta := ||a-b|| < \varepsilon$, then there exists $x \in A$ with:*

$$(a-\varepsilon)_+ = x^* b x\,.$$

Proof. (i): Assume without loss of generality, that $A \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Choose ε' such that $\delta < \varepsilon' < \varepsilon$. Combining Lemma 3.11 and Proposition 3.8 we see $(a - \varepsilon')_+ \precsim b$. By Proposition 3.10 (iii) and the definition of $\precsim$ there are $\gamma > 0$ and $x_1 \in A$ with

$$||(a-\varepsilon')_+ - x_1^*(b-2\gamma)_+ x_1|| < \varepsilon - \varepsilon',$$

and by Observation 3.12 we may replace $(b - 2\gamma)_+$ by $x_2^* h_{2\gamma}(b) x_2$ for some $x_2 \in A$ in the above inequality. Using Lemma 3.11 again we may choose $x_3 \in A$ with:

$$(a-\varepsilon)_+ = ((a-\varepsilon')_+ - (\varepsilon - \varepsilon'))_+ \le x_3^* x_1^* x_2^* h_{2\gamma}(b) x_2 x_1 x_3.$$

Now set $y := h_{2\gamma}(b)^{1/2} x_2 x_1 x_3$, and decompose y^* as $y^* = v|y^*|$ for some partial isometry $v \in \mathcal{B}(\mathcal{H})$ (see Theorem 3.1) and apply Lemma 3.3 to deduce $z := v|y^*|^{1/2} \in A$. Now $zh_\gamma(b) = z$ by $(*)$ below and hence:

$$(zh_\gamma(b)z^*)^2 = (zz^*)^2 = (y^*v^*)^2 = y^*v^*y^*v^* \overset{Th. 3.1}{=} y^*|y^*|v^* = y^*y \ge (a-\varepsilon)_+.$$

Now we apply Proposition 3.13, taking $(a-\varepsilon)_+^{1/2}$ as a, $(zh_\gamma(b)z^*)^2$ as b and $\alpha = \frac{1}{4}$, to obtain $x_4 \in A$ such that

$$(a-\varepsilon)_+ = \left((a-\varepsilon)_+^{1/2}\right)^*(a-\varepsilon)_+^{1/2} = x_4 z h_\gamma(b) z^* x_4.$$

Now choose $x_5 \in A$ with $x_5^* b x_5 = h_\gamma(b)$ and set $x := x_5 z^* x_4$ to obtain $(a-\varepsilon)_+ = x^* b x$.
Ad $(*)$: We show that already $|y^*|^{1/2} f_\gamma(b) = |y^*|^{1/2}$ holds. As

$$f_{2\gamma}(b) f_\gamma(b) = (f_{2\gamma} f_\gamma)(b) = f_{2\gamma}(b)\,,$$

the claim will follow, if we show, that $|y^*|^{1/2}$ is an element of the hereditary C^*-algebra generated by $f_{2\gamma}(b)$. This follows easily by the characterization of hereditary C^*-algebras (3.2) and

$$|y^*|^{1/2} = (yy^*)^{1/4} = \left(h_{2\gamma}(b)^{1/2} x_2 x_1 x_3 x_3^* x_1^* x_2^* h_{2\gamma}(b)^{1/2}\right)^{1/4},$$

finally completing the proof. □

The above Lemma gives a stronger relation than $\precsim$ between close elements in a C^*-algebra which is used in some papers also (see for example [LinZh]), but does not play an important role in our treatment (see Proposition 4.13. (i) though).

Proposition 3.15 (cf. [KiRr2, Proposition 2.6]). *Let $a, b \in A^+$; then the following conditions are equivalent:*

(i) $a \precsim b$

(ii) $\forall \varepsilon > 0 :\ (a-\varepsilon)_+ \precsim b$

(iii) $\forall \varepsilon > 0\ \exists \delta > 0 :\ (a-\varepsilon)_+ \precsim (b-\delta)_+$

(iv) $\forall \varepsilon > 0\ \exists \delta > 0\ \exists x \in A:\ (a-\varepsilon)_+ = x^*(b-\delta)_+ x$

Proof. (i)⇒(iv): Let $a \precsim b$, $\varepsilon > 0$ and $(x_n)_n \in A^{\mathbb{N}}$ with $x_n^* b x_n \to a$. Then

$$||x_n^*(b-\delta)_+ x_n - a|| \leq ||x_n||^2 ||(b-\delta)_+ - b|| + ||x_n^* b x_n - a||$$

and therefore we may choose $\delta > 0$ and $k \in \mathbb{N}$ with $||x_k^*(b-\delta)_+ x_k - a|| < \varepsilon$, and then use Lemma 3.14 (i) to find $x \in A$ with

$$(x_k x)^*(b-\delta)(x_k x) = (a-\varepsilon)_+ \,.$$

(iv)⇒(iii): This implication is obvious.
(iii)⇒(ii): Use $(b-\delta)_+ \precsim b$ (by Lemma 3.9 applied to φ_δ) and transitivity of $\precsim$.
(ii)⇒(i): For every $k \in \mathbb{N}$ choose a sequence $(x_n)_n^k \in A^{\mathbb{N}}$ from the definition of $\precsim$ with

$$||x_n^{k*} b x_n^k - (a - \frac{1}{k})_+|| \longrightarrow 0$$

as $n \to \infty$. Selecting for each $k \in \mathbb{N}$ some $n_k \in \mathbb{N}$ with $||x_{n_k}^{k*} b x_{n_k}^k - (a - \frac{1}{k})_+|| < \frac{1}{k}$ we obtain a sequence $(x_{n_k}^k)_k \in A^{\mathbb{N}}$ such that:

$$||x_{n_k}^{k*} b x_{n_k}^k - a|| \leq ||x_{n_k}^{k*} b x_{n_k}^k - (a - \frac{1}{k})_+|| + ||(a - \frac{1}{k})_+ - a|| \underset{k\to\infty}{\longrightarrow} 0 \,.$$

□

Observation 3.16. Observe that $(b-\delta)_+ = x^* b x$ for some $x \in A$ and therefore we may also take δ to be 0 in 3.15 (iv). In particular, if $a = p$ for some projection p in A, then there is some $x \in A$ such that $p = x^* b x$, because $(p - \frac{1}{2})_+ = \frac{1}{2} p$.

Definition 3.17. *Let $n \in \mathbb{N}$. For all $a, b \in M_n(A)^+$ we define an equivalence relation $\approx$ by setting $a \approx b$ if $a \precsim b$ and $b \precsim a$ hold.*

Remark 3.18. If $a, b \in A$ and $a \sim b$, then by definition there is some $x \in A$ such that $x^*x = a$ and $xx^* = b$, consequently we have $x^* b x = a^2$, implying $a \precsim a^2 \precsim b$ by Lemma 3.9. Applying this twice gives $a \approx b$.

Lemma 3.19 (cf. [KiRr2, Lemma 2.2]). *Let H be a hereditary C^*-subalgebra of A. Then for $a, b \in H$ we have $a \precsim b$ with respect to H if and only if $a \precsim b$ with respect to A.*

Proof. Assume $a, b \in H$ and $a \precsim b$ with respect to A. Then by Lemma 3.9 we have $a^{1/2} \precsim b^2$, and by definition of $\precsim$ there is a sequence $(x_n)_n \in A^{\mathbb{N}}$ such that $x_n^* b^2 x_n \to a^{1/2}$. We define $y_n := b^{1/2} x_n a^{1/4}$ to obtain an element of H by the characterization of hereditary C^*-algebras (Theorem 3.2) and deduce:

$$y_n b y_n = a^{1/4} x_n^* b^{1/2} b b^{1/2} x_n a^{1/4} \to a^{1/4} a^{1/2} a^{1/4} = a \,,$$

giving $a \precsim b$ with respect to H, as claimed. □

Lemma 3.20. *Let a be a positive element of A. Then for all $b \in \overline{aAa}$ we have $b \precsim a$.*

Proof. We have $b^{1/2} \in \overline{aAa}$, hence there is a sequence $(x_n)_n$ such that $ax_na \to b^{1/2}$. Set $y_n := x_na^2x_n^*$, observe that the y_n are positive and

$$ay_na = ax_na\,(ax_na)^* \to b\,.$$

Further, $ay_na \leq ||y_n||a^2 \precsim a$, therefore Proposition 3.8 implies $ay_na \precsim a$. As for all $\varepsilon > 0$ we may choose $n \in \mathbb{N}$ with $||b - ay_na|| \leq \varepsilon$ and deduce $(b-\varepsilon)_+ \precsim ay_na \precsim a$ by Lemma 3.11, we have proved the lemma by Proposition 3.15. □

3.2 Finiteness conditions on elements of C^*-algebras

Observe that for any $a, b \in A^+$ we have $a \oplus 0 \leq a \oplus b$, hence $a \precsim a \oplus b$, by combining Proposition 3.6 (i), Remark 3.18 and Proposition 3.8. The following definitions are taken from [KiRr2].

Definition 3.21 (Finite and infinite). *Let $a \in A^+$; then a is called infinite if there exists a nonzero $b \in A^+$ such that $a \oplus b \approx a$. If $a \oplus b \precsim a$ does not hold for any nonzero $b \in A^+$, then a is called finite.*

Definition 3.22 (Properly infinite). *Let $a \in A^+$; then a is called properly infinite if $a \oplus a \approx a$.*

Remark 3.23. Note that if $a, b \in A_+$ and $a \approx b$, then a is (properly) infinite if and only if b is.

Remark 3.24. We will now take a look at how $\precsim$ and finiteness conditions behave with respect to homomorphisms.
The right notion of a morphism preserving the order $\leq$ of a C^*-algebra is a positive map. In particular, any *-homomorphism is positive, hence it might be interesting to know, what the order morphisms with respect to $\precsim$ are. If $a \precsim b$ and $(x_n)_n \in A^{\mathbb{N}}$ such that $x_n^*bx_n \to a$ as $n \to \infty$, then $\varphi(x_n)^*\varphi(b)\varphi(x_n) \to \varphi(a)$. Consequently, *-homomorphisms preserve $\precsim$; as we use that φ preserves * and is continuous (already as a consequence of being a morphism of C^*-algebras), it seems that we use the full arsenal of *-homomorphism properties.
But in fact, the following property may be added to the list in Proposition 3.15:

$$\text{(v)} \qquad \exists (x_n)_n, (y_n)_n \in A^{\mathbb{N}} :\ x_nby_n \to a\,.$$

This condition is obviously implied by $a \precsim b$; conversely, if $x_nby_n \to a$ for sequences $(x_n)_n, (y_n)_n \in A^{\mathbb{N}}$, then setting $v_n := x_nb^{1/2}$ and $w_n := b^{1/2}y_n$ we have:

$$v_nw_nw_n^*v_n^* = x_nb\,y_ny_n^*b\,x_n^* \to aa^* = a^2.$$

For $\varepsilon > 0$ choose $n \in \mathbb{N}$ with $||v_n w_n w_n^* v_n^* - a^2|| < \varepsilon$. Then Lemma 3.14 implies

$$(a^2 - \varepsilon) \precsim v_n w_n w_n^* v_n^* \leq ||w_n||^2 x_n b x_n^* \precsim b \,,$$

hence $a \sim a^2 \precsim b$ holds.
The condition (v) is then preserved under any continuous, multiplicative map between C^*-algebras.
Now if $a \in A^+$ and $b \in A^+$ such that $a \oplus b \precsim a$, then $\varphi(b) \in A^+$ and $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b) \precsim \varphi(a)$ by the above arguments. Applying this in the case $b = a$ we obtain that properly infinite elements map to such under *-homomorphisms, but if a is infinite and $b \in A_+$ is nonzero such that $a \oplus b \approx a$, then b may map to zero and hence the image of a might still be finite.

Lemma 3.25 (cf. [KiRr2, Proposition 3.3]). *Let $0 \neq a \in A^+$. Then the following properties are equivalent:*

(i) *a is properly infinite.*

(ii) *For all $\varepsilon > 0$: $(a - \varepsilon)_+ \oplus (a - \varepsilon)_+ \precsim a$.*

(iii) *For all $\varepsilon > 0$ exist $a_1, a_2 \in \overline{aAa}$ such that $a_1 \perp a_2$ and $(a - \varepsilon)_+ \precsim a_j$.*

(iv) *There are sequences $(x_n)_n, (y_n)_n \in \overline{aAa}^{\mathbb{N}}$ such that: $x_n^* x_n \to a$, $y_n^* y_n \to a$ and $x_n^* y_n \to 0$.*

(v) *For all $\varepsilon > 0$ there are $x, y \in \overline{aAa}$ such that $x^* x = (a - \varepsilon)_+ = y^* y$ and $xx^* \perp yy^*$.*

Proof. Obviously (i) and (ii) are equivalent by Proposition 3.15 and Lemma 2.8.
We show (i)$\Rightarrow$(v). Let $n \in \mathbb{N}$ and set $\varepsilon := \frac{1}{n}$. By Observation 3.16 applied inside $M_2(A)$ we may choose $z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in M_2(A)$ with

$$\begin{pmatrix} (a - \varepsilon)_+ & 0 \\ 0 & (a - \varepsilon)_+ \end{pmatrix} = z^*(a \oplus 0) z = \begin{pmatrix} z_1^* a z_1 & z_1^* a z_2 \\ z_2^* a z_1 & z_2^* a z_2 \end{pmatrix}$$

giving, in particular, $z_1^* a z_2 = 0$. Setting $x_n := a^{1/2} z_1$ and $y_n := a^{1/2} z_2$ we obtain (v).
(v)$\Rightarrow$ (iv): Choose x_n, y_n as in (v) such that $x_n^* x_n = (a - \frac{1}{n})_+ = y_n^* y_n$, obtaining

$$\lim_{n \to \infty} x_n^* x_n = \lim_{n \to \infty} (a - \frac{1}{n})_+ = a$$

by 3.10 (iii). It now suffices to show $x_n^* y_n = 0$ for all $n \in \mathbb{N}$. We already have

$$|x_n^*| = (x_n x_n^*)^{1/2} \perp (y_n y_n^*)^{1/2} = |y_n^*| \,.$$

Now take polar decompositions $x_n = |x_n^*| v_n$ and $y_n = |y_n^*| w_n$, where v_n and w_n are partial isometries (by Theorem 3.1). It follows

$$x_n^* y_n = v_n^* |x_n^*| \, |y_n^*| w_n = 0$$

as desired.
(iv)$\Rightarrow$ (i): Let $(x_n)_n$, $(y_n)_n$ be sequences in $\overline{aAa}$ such that $x_n^*x_n \to a$, $y_n^*y_n \to a$ and $x_n^*y_n \to 0$. Then $x_n, y_n \in \overline{a^{1/2}A}$ and we may thus choose, for every $N \in \mathbb{N}$ sequences $(s_n^N)_n, (t_n^N)_n$ with:

$$a^{1/2}s_n^N \to x_N, \qquad a^{1/2}t_n^N \to y_N .$$

Now choose $n_N \in \mathbb{N}$ with

$$||a^{1/2}s_{n_N}^N - x_N|| < \frac{1}{N}, \qquad ||a^{1/2}t_{n_N}^N - y_N|| < \frac{1}{N} ,$$

implying, where $u_N := s_{n_N}^N$, $v_N := t_{n_N}^N$:

$$||a^{1/2}u_N - x_N|| \to 0, \qquad ||a^{1/2}v_N - y_N|| \to 0 .$$

Now $||x_n||^2 = ||x_n^*x_n|| \leq M$ for some $M \in \mathbb{R}^+$, as $(x_n^*x_n)_n$ is a convergent sequence, and therefore $(a^{1/2}u_n)_n$ is bounded also; similarly $(a^{1/2}v_n)_n$ is bounded. We thus obtain

$$\begin{aligned}
||u_n^*au_n - a|| \leq& ||u_n^*au_n - x_n^*x_n|| + ||x_n^*x_n - a|| \\
\leq& ||u_n^*au_n - u_n^*a^{1/2}x_n + u_n^*a^{1/2}x_n - x^*x|| + ||x_n^*x_n - a|| \\
\leq& ||a^{1/2}u_n||\,||a^{1/2}s_n - x|| + ||x_n||\,||s_n^*a^{1/2} - x_n^*|| + ||x_n^*x_n - a|| \to 0
\end{aligned}$$

and therefore $u_nau_n \to a$; similarly $v_nav_n \to a$. Furthermore we have:

$$||u_n^*av_n - x_n^*y_n|| \leq ||u_n^*a^{1/2}||\,||a^{1/2}v_n - y_n|| + ||y_n||\,||u_n^*a^{1/2} - x_n^*|| ,$$

which again tends to 0. As $x_n^*y_n \to 0$ we obtain $u_n^*av_n \to 0$. $\square$

Lemma 3.26. *Let $a \in A^+$ be a properly infinite, non-zero element. Then $b \precsim a$ for all b in the closed two sided ideal $\overline{AaA}$ generated by a.*

Proof. Let $\varepsilon > 0$. As the elements of the form $\sum_{i=0}^n x_i^a y_i$ form the algebraic ideal generated by a, we may choose $x_1, \dots, x_n, y_1 \dots, y_n \in A$ with

$$||b - \sum_{i=1}^n x_iay_i|| < \varepsilon .$$

Therefore

$$(b-\varepsilon)_+ \precsim \sum_{i=1}^n x_iay_i = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} (a \otimes 1_n) \begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{pmatrix} \precsim a \otimes 1_n \precsim a,$$

using Lem. 3.14 (i) and 3.15, showing that $b \precsim a$ holds again by Proposition 3.15. $\square$

Remark 3.27. If B is abelian and $b \in B^+$ properly infinite, then we may choose $x, y \in B$ as in (v) and deduce that $||(xx^*)||^2 = ||(xx^*)(yy^*)|| = 0$, and therefore $b = 0$. In particular, a properly infinite element maps to a properly infinite element under any quotient morphism, as noted in Remark 3.24, and will therefore map to zero in any abelian quotient.

3.3 Comparison theory and finiteness conditions for projections

We now investigate how the above comparison theory applies to projections in C^*-algebras. The relation $\sim$ restricted to the set of projections is the classical Murray-von Neumann equivalence. Recall that a sub-projection q of p is a projection with $q \leq p$ and is called a proper sub-projection if in addition $p \neq q$.

Lemma 3.28. *Let p be a projection in A and $v \in A$ such that $v^*v = p$. Then $v = vv^*v$ and vv^* is a projection.*

Proof. We have

$$0 \leq v^*(1 - vv^*)(1 - vv^*)v = v^*v - v^*vv^*v - v^*vv^*v + v^*vv^*vv^*v = p - p - p + p = 0$$

and hence $(1 - vv^*)v = 0$, implying $v = vp$. We also obtain that vv^* is a projection, as $vv^*vv^* = vv^*$. □

The next proposition shows that $\precsim$ extends the classical comparison theory by Murray and von Neumann for projections to arbitrary positive elements.

Proposition 3.29 (cf. [Rr2, Proposition 2.1]). *Let $p, q \in \mathcal{P}(A)$. Then $p \precsim q$ if and only if p is equivalent to a subprojection of q.*

Proof. $\Rightarrow$: For any projection p we have $\sigma(p) \subseteq \{0, 1\}$ by the commutative Gel'fand theorem (in fact, for normal elements this is equivalent to being a projection), therefore we have for any $\varepsilon \in (0, 1)$ that $(p - \varepsilon)_+ = \varepsilon p$. Application of Lemma 3.14 gives some $a \in A$ with $\varepsilon p = a^*qa$ and setting $y := \varepsilon^{-1/2}a$ we get $p = y^*qy$. Defining $v := y^*q$ we have $vv^* = p$ and consequently $v^*v = qyy^*q$ is a projection by Lemma 3.28. As v^*v commutes with q, [M, Theorem 2.3.2] implies $v^*v \leq q$.
$\Leftarrow$: Let $v \in A$ with $v^*v = p$ and $vv^* \leq q$. Then $p = v^*vv^*v \leq v^*qv$ and Proposition 3.8 implies $p \precsim v^*qv \precsim q$. □

Proposition 3.30 (cf. [KiRr2, Lemma 3.1]). *A projection $p \in A$ is infinite if and only if it is Murray-von Neumann equivalent to a proper sub-projection of itself.*

Proof. $\Rightarrow$: Assume that there is $a \in A_+$ such that $p \oplus a \precsim p$. Let $\varepsilon \in (0, 1)$; then by Proposition 3.10 (vii) and Proposition 3.15 there is $x \in A$ such that

$$(1 - \varepsilon)p \oplus (a - \varepsilon)_+ = (p - \varepsilon)_+ \oplus (a - \varepsilon)_+ = ((p \oplus a) - \varepsilon)_+ = x^*(1 - \varepsilon)px$$

(see Observation 3.16). Set $b := (1 - \varepsilon)^{-1}(a - \varepsilon)_+$; then we have shown that there are $x_1, x_2 \in A$ with:

$$\begin{pmatrix} p & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} p \begin{pmatrix} x_1 & x_2 \end{pmatrix} = \begin{pmatrix} x_1^*px_1 & x_1^*px_2 \\ x_2^*px_1 & x_2^*px_2 \end{pmatrix}.$$

Setting $v := px_1$ we have $v^*v = p$, and from $vv^*p = vv^*$ we get $q := vv^* \leq q$. It remains to show $q \neq p$. But otherwise:

$$b = x_2^* p x_2 = x_2^* p x_2 = x_2^* v v^* x_2 = x_2^* p x_1 x_1^* p x_2 = 0 \,.$$

$\Leftarrow$: Let p_0 be a proper subprojection of p such that $p_0 \sim p$. Then $0 \neq p - p_0 \geq 0$, $p - p_0 \perp p_0$, and

$$p \oplus (p - p_0) \overset{Prop.\ 3.6(iii)}{\sim} p_0 \oplus (p - p_0) \overset{Prop.\ 3.6(iv)}{\sim} p_0 + p - p_0 = p \,.$$

Remark 3.18 shows $p \oplus (p - p_0) \precsim p$. □

Proposition 3.31. *A projection p in A is properly infinite if and only if there are pairwise orthogonal projections $e, f \in A$ such that $e \sim p \sim f$ and $e \leq p$, $f \leq p$.*

Proof. $\Rightarrow$: Assume that p is properly infinite. Then by Proposition 3.29 there are $v_1, v_2 \in A$ such that

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} v_1^* v_1 & v_1^* v_2 \\ v_2^* v_1 & v_2^* v_2 \end{pmatrix} = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} \sim \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} = v_1 v_1^* + v_2 v_2^* \leq p \,,$$

whereas we may set $e := v_1 v_1^*$ and $f := v_2 v_2^*$ to obtain orthogonal projections (see Lemma 3.28) that are equivalent to p and such that $e + f = p$. Consequently, $e \leq e + f \leq p$ and $f \leq e + f \leq p$.
$\Leftarrow$: Let $e, f \in \mathcal{P}(A)$ such that $e \perp f$ and $e \sim p \sim f$, $e \leq p$, $f \leq p$. We will show that Lemma 3.25 (v) holds. We may assume $1 > \varepsilon > 0$ and use that in this case $\varepsilon p = (a - \varepsilon)_+$; consequently it suffices to show $p = x^*x = y^*y$ and $xx^* \perp yy^*$ for some $x, y \in pAp$. But this is obvious, choosing x and y as the partial isometries implementing the equivalence between p and e, f respectively, and observing that $e, f \in pAp$. □

In the sequel we will utilize the above characterization for finiteness conditions when dealing with projections rather than the more general definitions for arbitrary positive elements.
As noted in Remark 3.23 infinity of positive elements is preserved under passing to equivalent elements. We give another proof of this fact for projections using the characterization of infiniteness via subprojections:

Proposition 3.32. *Let $p, q \in \mathcal{P}(A)$ with $p \sim q$. Then p is infinite if and only if q is.*

Proof. We have $v, w \in A$ such that

$$\begin{aligned} p &= v^*v & q &= vv^* \\ p &= w^*w & p_0 &:= ww^* < p, \end{aligned}$$

For $q_0 := vp_0v^*$ we have

$$q_0^2 = vp_0v^*vp_0v^* = vp_0pp_0v^* = vp_0v^*,$$

and hence q_0 is a projection, being obviously self adjoint. Also

$$qq_0 = qvp_0v^* \overset{Lem.\ 3.28}{=} vp_0v^* = q_0$$

and therefore $q_0 \leq q$. Further $q_0 \neq q$ as $q_0 = q$ would imply:

$$p_0 = v^*vp_0v^*v = v^*qv = p \ .$$

Now let $x := p_0v^*$. Then we have

$$x^*x = vp_0v^* = q_0 \qquad\qquad xx^* = p_0v^*vp_0 = p_0pp_0 = p_0 \sim p \sim q \ ,$$

implying $q_0 \sim q$ by transitivity of $\sim$. $\square$

3.4 Dimension functions

We will need the following material on dimension functions and related definitions in order to prove basic properties of so called purely infinite C^*-algebras to be introduced in the next section.
Recall that the Pedersen ideal $\mathrm{Ped}(A)$ of A is the intersection over all dense algebraic two sided ideals of A; as it is a dense algebraic two sided ideal itself, it is minimal with respect to this property. Proofs of these facts can essentially be found in the section on the minimal dense ideal in [Ped2].

Definition 3.33 (Dimension Function). *Let A be a C^*-algebra and I an algebraic ideal in A. Then a dimension function on A with domain I is a function $d\colon \bigcup_{n\in\mathbb{N}} M_n(I)^+ \to \mathbb{R}$ subject to the following conditions:*

(i) $d(a \oplus b) = d(a) + d(b)$,

(ii) $d(a) \leq d(b)$ *if* $a \precsim b$.

A dimension function is called faithful if for all $a \in \mathrm{Ped}(A)^+\setminus\{0\}\cap I$ we have $d(a) \neq 0$.

For every *-homomorphism $\varphi\colon A \to B$ between C^*-algebras and every dimension function d on B we get a dimension function φ^*d on A by setting $\varphi^*d(a) := d(\varphi(a))$.

Proposition 3.34. *Let A be a C^*-algebra and d a nontrivial dimension function on A. We have*

(i) $N := \{x \in A \mid d(x^*x) = 0\}$ *is an algebraic ideal in A,*

(ii) *If A is algebraically simple, then d is faithful.*

Proof. (i) For $a \in N$, $x \in A$ we have $x^*a^*ax \precsim a^*a$ and hence:

$$d((ax)^*ax) \leq d(a^*a) = 0.$$

That N is closed under addition and scalar multiplication follows, as $a + b \precsim a \oplus b$ for all $a, b \in A$ and $\bar{\lambda}a^*a \precsim a^*a$.
(ii) is a consequence of (i). $\square$

Theorem 3.35 (cf. [BlaCu, Theorem 1.2]). *Let A be a simple C^*-algebra such that $A \otimes \mathcal{K}(\mathcal{H})$ admits no non-trivial dimension function defined on its Pedersen ideal. Then $A \otimes \mathcal{K}(\mathcal{H})$ contains an infinite projection.*

4 Finiteness conditions on C^*-algebras

4.1 Finite and infinite C^*-algebras

Definition 4.1 (Finite and infinite C^*-algebras). *A unital C^*-algebra A is called finite if its unit is finite, otherwise it is called infinite.*

Proposition 4.2. *Let A be a unital C^*-algebra, p a projection in A. Then:*

(i) *p is finite iff pAp is finite,*

(ii) *A is finite iff every isometry in A is unitary*

(iii) *A is infinite iff the unital Toeplitz algebra embeds unitally into A.*

Proof. (i): That p is infinite if pAp is follows directly from the definition (noting that p is a unit for pAp). Now let p be an infinite projection in A. Then there is, by Proposition 3.30, a proper subprojection $q \in A$ of p which is equivalent to p, hence a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$. It suffices to show that $v, q \in pAp$; by we then have $p = pqp \in pAp$ and $v = vp = pqvp \in pAp$, implying that p is infinite.
(ii): Now let A be finite, v an isometry in A. Then

$$1_A = v^*v \sim vv^* \leq 1_A$$

and by finiteness of 1_A we have $vv^* = 1$ (in fact, the generalized Murray-van Neumann equivalence class of 1_A in $K_0(A)$ is the singleton 1_A). Conversely, if every isometry in A is unitary and $v^*v = 1$, then $vv^* = 1$ and therefore 1_A is finite.
(iii): The Toeplitz algebra is the universal C^*-algebra generated by a nonunitary isometry (cf. [M, 3.5.18]), whence an algebra containing it up to isomorphism is infinite by (ii). If A is infinite, then there is a nonunitary isometry in A by (ii), and therefore there is an embedding of the Toeplitz algebra in A. □

Proposition 4.3. *Let A be a unital C^*-algebra. Then A is finite iff every projection in A is finite.*

Proof. Let $p, q \in \mathcal{P}(A)$ with $p \sim q$. Choose $v \in A$ such that $p = v^*v$ and $q = vv^*$. Then for $u := v + (1-p)$ we get, applying Lemma 3.28 repeatedly, that

$$u^*u = v^*v + v^*(1-p) + (1-p)v + (1-p) = p + v^* - v^*qp + v - pqv = 1$$

and

$$uu^* = vv^* + (1-p) + (1-p)v^* + v(1-p) = 1 + (q-p),$$

as $uu^* = 1$ by Proposition 4.2 we deduce $p = q$.
The converse is obvious. $\square$

4.2 Properly infinite C^*-algebras

Definition 4.4 (properly infinite). *A unital C^*-algebra A is called properly infinite if its unit is a properly infinite projection.*

Lemma 4.5. *Let A be a properly infinite C^*-algebra. Then A contains a sequence of isometries $(s_n)_{n\in\mathbb{N}}$ with mutually orthogonal range projections.*

Proof. By the definition of properly infinite there are isometries t_1, t_2 in A with $1_A \sim t_1t_1^* \perp t_2t_2^* \sim 1_A$. As in Observation 4.14 we have $t_1^* \perp t_2$ and setting $s_n := t_2^{n-1}t_1$ for all $n \in \mathbb{N}$ we have $s_n^*s_m = t_1^*(t_2^*)^{n-1}t_2^{m-1}t_1 = \delta_{nm}$ for all $m, n \in \mathbb{N}$. $\square$

4.3 Purely infinite C^*-algebras

In this section, we introduce the notion of purely infinite for C^*-algebras. This notion is stronger than being properly infinite; we will spend a considerable amount of time investigating equivalent conditions to being purely infinite. In the first section, conditions that hold even for-non simple C^*-algebras are presented; the setting is somewhat simplified in the case of simple purely infinite C^*-algebras, the which will be treated in the second section.

4.3.1 Non-simple purely infinite C^*-algebras

Definition 4.6 (Purely Infinite). *Consider the following two conditions on a C^*-algebra A:*
Condition I: Every nonzero positive element in A is properly infinite.
Condition II: A has no characters and for all $a, b \in A^+$ with $a \in \overline{AbA}$: $a \precsim b$.
We will call A purely infinite if A satisfies Condition I.

The main task is to prove that the above condition I is equivalent to the less intuitive condition II.
Having developed the necessary comparison theoretical techniques, it is easy to see that Condition I implies Condition II:
If A satisfies condition I, then A can not have any non-zero abelian quotients by Remark 3.24. If now $\varphi\colon A \to \mathbb{C}$ were a character on A, then we would have $A/\operatorname{Ker}(\varphi) \simeq \mathbb{C}$, which would be a contradiction. As the second part of condition II is implied by Lemma 3.26, we see that Condition I implies Condition II.
One has to work harder to see that Condition II implies condition I. We only outline the proof, the rigorous one is given in [KiRr2].

One shows first that, given a short exact sequence of C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

such that all nonzero positive elements in I and B are properly infinite, all nonzero positive elements in A have to be properly infinite also.
Given a C^*-algebra satisfying Condition II, one considers the set $\mathcal{I}$ of those ideals in A such that all their nonzero elements are properly infinite. For any two ideals I and J in this set, one therefore has a short exact sequence

$$0 \longrightarrow I \longrightarrow I + J \longrightarrow (I + J)/I \longrightarrow 0$$

of C^*-algebras, and all nonzero positive elements in I are properly infinite by hypothesis. As $(I + J)/I \cong J/(I \cap J)$, and being properly infinite is preserved by *-homomorphisms, all elements in $(I + J)/I$ are properly infinite, we may now apply the result on short exact sequences above, to see that $\mathcal{I}$ is upwards directed. An application of Lemma 3.25 gives that the union I_0 of all ideals in $\mathcal{I}$ is an element of $\mathcal{I}$ (one could also use Zorn's Lemma to obtain a maximal element in this set).
To finish the proof, one assumes $I_0 \neq A$. Then a version of Glimm's Lemma shows that $M_2\big(C_0((0,1])\big)$ embeds into the quotient A/I_0, and every nonzero positive element in the ideal generated by it's image is properly infinite. Another application of the result on short exact sequences gives a contradiction to the maximality of I_0.

Proposition 4.7. *Let A be a purely infinite C^*-algebra and H an hereditary C^*-subalgebra. Then H is purely infinite.*

Proof. For every element $a \in H$ we have $a \oplus a \precsim a$ in A by the definition of properly infinite. Now H is hereditary and therefore this statement also holds with respect to H by Lemma 3.19. $\square$

4.3.2 Simple purely infinite C^*-algebras

In this section we will consider simple purely infinite C^*-algebras. From the last section we already know that these are C^*-algebras containing only properly infinite positive elements, and this is equivalent to Condition II above, i.e., $a \approx b$ for any two nonzero $a, b \in A^+$ in the case of simple C^*-algebras.

Observation 4.8. Let B be a stable C^*-algebra , i.e., $B \cong \mathcal{K}(\mathcal{H}) \otimes B$. Then B contains a sequence $(a_n)_n$ of positive, pairwise orthogonal nonzero elements such that $a_j \sim a_i$ for all $i, j \in \mathbb{N}$.
One may construct this sequence as follows: Take an isomorphism $\varphi\colon \mathcal{K}(\mathcal{H}) \otimes B \to B$, where $\mathcal{H}$ is some separable Hilbert space. Let e_{ij} be a set of standard matrix units for $\mathcal{K}(\mathcal{H})$ and choose some $h \in B^+$. Define

$$a_i := \varphi(e_{ii} \otimes h)\,.$$

Then $a_i a_j = \varphi(e_{ii}e_{jj} \otimes h) = 0$ for $i \neq j$ and

$$a_i = \varphi(e_{ij} \otimes h^2)\varphi(e_{ji} \otimes h) \sim \varphi(e_{ji} \otimes h)\varphi(e_{ij} \otimes h) = a_j,$$

implying pairwise equivalence of the a_i.

Lemma 4.9. *Let A be a purely infinite C^*-algebra. Then A has no nonzero dimension function. If A is simple, then $A \otimes \mathcal{K}(\mathcal{H})$ does not admit any dimension function on its Pedersen ideal.*

Proof. Let d be a dimension function on A; for every $a \in A$, which lies in the domain of d and is therefore positive, we have $a \oplus a \precsim a$, yielding $d(a)+d(a) = d(a \oplus a) \leq d(a)$, in turn entailing $a = 0$ and triviality of d.
Note that A embeds into $A \otimes \mathcal{K}(\mathcal{H})$ via $\varphi = \iota_{e_{11}}$, sending a to $a \otimes e_{11}$ (cf. Lemma 2.7). Assume d is a dimension function on $\mathrm{Ped}(A \otimes \mathcal{K}(\mathcal{H}))$. We will show that the dimension function $\varphi^* d = d\varphi$ has nontrivial domain, which is a contradiction to the first part of the lemma. The finite rank operators are dense in $\mathcal{K}(\mathcal{H})$ ([M, Theorem 2.4.5]), hence the algebraic ideal generated by $A \otimes e_{11}$ is dense in $A \otimes \mathcal{K}(\mathcal{H})$, and consequently contains the Pedersen ideal of $A \otimes \mathcal{K}(\mathcal{H})$. If $N = \{x \in A \otimes \mathcal{K}(\mathcal{H}) \mid d(x^*x) = 0\}$ contained $A \otimes e_{11}$, then the Pedersen ideal of $A \otimes \mathcal{K}(\mathcal{H})$ would be a subset of N by Proposition 3.34, thence d would have to be zero. □

Lemma 4.10. *Let A be a simple C^*-algebra such that for every $a \in \mathrm{Ped}(A)^+$ the hereditary C^*-subalgebra $\overline{aAa}$ contains a nontrivial stable C^*-subalgebra. Then there exists no non-zero dimension function on* $\mathrm{Ped}(A)$.

Proof. Let $a \in P(A)^+$. Then there is a sequence $(a_n)_{n \in \mathbb{N}}$ as in Observation 4.8, and $\sum\limits_{i=1}^{n} a_i \precsim a$ for all $n \in \mathbb{N}$ by Lemma 3.20. Now let d be a dimension function on $P(A)$, then we have

$$nd(a) = \sum_{i=1}^{n} d(a_i) = d(\sum_{i=1}^{n} a_i) \leq d(a),$$

and henceforth $d(a)$ must be zero. □

Lemma 4.11. *Let A be a simple C^*-algebra containing an infinite projection p. Then for any $a \in A^+$ there is some $x \in A$ with $x^*px = a$.*

Proof. Find $v \in A$ with $vv^* = p$ and $v^*v \leq p$. By the last line of the proof of [BlaCu, Proposition 2.6] applied to v we may choose $x \in A$ such that $xx^* = a$ and $px^*x = x^*x$. Then from

$$(px^* - x^*)(px^* - x^*)^* = (p-1)x^*x(p-1) = 0$$

we deduce $px^* = x^*$ and

$$xpx^* = xx^* = a.$$

□

We need one more definition for the proof of the next proposition:

Definition 4.12. *Let a be an element in a C^*-algebra A. Then x will be called scaling if $a^*a \neq aa^*$ and $(a^*a)(aa^*) = aa^*$.*

For example, if p is an infinite projection in A and q a proper subprojection of p which is equivalent to p, then the partial isometry v such that $v^*v = p$ and $vv^* = q$ is a scaling element.

Proposition 4.13. *Let A be a simple C^*-algebra. Then A is purely infinite if and only if one of the following conditions holds:*

(i) *A is not isomorphic to $\mathbb{C}$ and for every pair of non-zero elements $a, b \in A^+$ there exists $x \in A$ such that $a = x^*bx$,*

(ii) *Every non-zero hereditary C^*-subalgebra of A contains an infinite projection*

(iii) *Every non-zero hereditary C^*-subalgebra of A contains a non-trivial stable C^*-subalgebra*

Proof. We show: A purely infinite $\Rightarrow$ (ii)$\Rightarrow$(iii) $\Rightarrow$ (i) $\Rightarrow$ A purely infinite.
Let A be a purely infinite simple C^*-algebra. By Lemma 4.9 there is no dimension function on the Pedersen ideal of A; by 3.35 we obtain an infinite projection $p \in A \otimes \mathcal{K}(\mathcal{H})$. Let B be a hereditary C^*-subalgebra of A, then B is also a hereditary C^*-subalgebra of $A \otimes \mathcal{K}(\mathcal{H})$ (by slight abuse of notation), and by simplicity, for any $b \in B^+$ the closed two sided ideal of $A \otimes \mathcal{K}(\mathcal{H})$ generated by b contains p. Now b is properly infinite, hence by Lemma 3.26 we deduce $p \precsim b$, hence by Proposition 3.15 there is some $x \in A \otimes \mathcal{K}(\mathcal{H})$ such that $p = x^*bx$. Setting $v := x^*b^{1/2}$ we obtain an infinite projection $v^*v \in (A \otimes \mathcal{K}(\mathcal{H}))b(A \otimes \mathcal{K}(\mathcal{H})) \subseteq B$.
Assume (ii) holds. Let H be a hereditary C^*-subalgebra of A and $p \in H$ an infinite projection. Then $pAp \subseteq H$ by [M, Corollary 3.2.4] and by Proposition 4.2 the Toeplitz algebra, and hence the compact operators on some separable infinite dimensional Hilbert space, embed into pAp, showing that pAp is a stable C^*-subalgebra of A.
Assume (iii) holds and let $a, b \in A^+$. Then every hereditary C^*-subalgebra H of $\overline{aAa}$ is also a hereditary C^*-subalgebra of A and therefore the hypothesis holds also for $\overline{aAa}$, as it is simple by [M, Theorem 3.2.8]. Consequently we may apply Lemma 4.10 combined with [BlaCu, Lemma 4.5] to see that $\overline{aAa}$ contains a scaling element. Now applying [BlaCu, Theorem 3.1] gives an infinite projection $p \in \overline{aAa}$. It follows from Lemma 3.20 and Observation 3.16 that there is some $x \in A$ with $p = x^*ax$, further we may apply Lemma 4.11 to choose $y \in A$ with $b = y^*py$ and conclude:

$$b = y^*py = y^*x^*axy.$$

Assume (i) holds. Then all elements are $\approx$-equivalent in A and it suffices to show the existence of a properly infinite element in A by Remark 3.23. Applying [RrLL, Lemma 7.1.4], we deduce that A has a maximal abelian C^*-subalgebra M not isomorphic to $\mathbb{C}$. Therefore $M \cong C_0(X)$ via an isomorphism $\varphi : C_0(X) \to M$, where X is a locally compact Hausdorff space with $|X| > 1$. We may thus choose nonzero positive functions $f, g \in C_0(X)$ such that $fg = 0 = gf$ and we have $\varphi(f) \approx a$ and $\varphi(g) \approx a$ for any $a \in A^+$. It follows:

$$a \oplus a \precsim \varphi(f) \oplus \varphi(g) \precsim \varphi(f + g) \precsim a,$$

the which implies that a, and consequently every positive element in A, is properly infinite. $\square$

4.4 Cuntz

4.4.1 The Cuntz algebras $\mathcal{O}_n$

In this section we will introduce the so called Cuntz algebras, which will play a "dominant" role in the proof of the exact embedding theorem. Apart from that, they are an example of purely infinite C^*-algebras; in fact, they were the first example of a purely infinite C^*-algebra. The algebras $\mathcal{O}_n$ were introduced by J. Cuntz in "Simple C^*-algebras generated by Isometries" ([Cu1]) and are a somewhat natural generalization of the Toeplitz algebra; later on, he defined in [Cu2] the notion of purely infinite and proved that the algebras $\mathcal{O}_n$ have this property.
We start with a concrete realization of $\mathcal{O}_n$. Let $n \in \mathbb{N}$ and set $\mathcal{H}_k := l^2(\mathbb{N})$ for all $k = 1, \dots, n$, define $\mathcal{H} := \bigoplus_{k=1}^n \mathcal{H}_k$. We denote the canonical inclusion of $\mathcal{H}_k$ into $\mathcal{H}$ by ι_k and the projection from $\mathcal{H} \to \mathcal{H}_k$ by π_k. Then $\sum_{k=1}^n \iota_k \pi_k = \mathrm{id}_{\mathcal{H}}$ and $\pi_k \iota_l = \delta_{k,l}$ ($\bigoplus$ is a categorical bi-product). Now let $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}_k$, then

$$\langle \xi \mid \iota_k(\eta) \rangle = \sum_{j=1}^n \langle \pi_j(\xi) \mid \pi_j \, \iota_k(\eta) \rangle_{\mathcal{H}_j} = \langle \pi_k(\xi) \mid \eta \rangle_{\mathcal{H}} \ ,$$

that is, the adjoint of the k-th projection is the k-th inclusion.
Because $\mathcal{H}$ and $l^2(\mathbb{N})$ are isomorphic as Hilbert spaces, there is an isometric surjection, hence a unitary operator, $u_k : \mathcal{H} \to \mathcal{H}_k$. Set $s_k := \iota_k \, u_k$. Then

$$s_k^* s_l = u_k^* \, \pi_k \, \iota_l \, u_l = \delta_{k,l} \ ,$$

and further

$$\sum_{j=1}^n s_k s_k^* = \sum_{j=1}^n \iota_k \, u_k \, u_k^* \, \pi_k = \sum_{j=1}^n \iota_k \, \pi_k = \mathrm{id}_{\mathcal{H}} \ .$$

It is easy to see that one may also realize this algebra as a kind of shift operator an a sum of Hilbert spaces.
For a unital C^*-algebra A, $n \in \mathbb{N}_{\geq 2}$ and $s_1, \dots, s_n$ isometries in A, we say that $(s_i)_{1 \leq i \leq n}$ satisfies the Cuntz relation (or $\mathcal{O}_n$-relation), if

$$\sum_{i=1}^n s_i s_i^* = 1.$$

Observation 4.14. Given $n \in \mathbb{N}$ $(s_i)_{1 \leq i \leq n}$ as above, we may use Lemma 2.5 to deduce

$$(s_i s_i^*)(s_k s_k^*) = \delta_{i,k} s_i s_i^* \ .$$

As isometries are left invertible and coisometries are right invertible, we immediately deduce:

$$s_i^* s_k = \delta_{i,k}.$$

An infinite sequence of isometries $(s_i)_{i\in\mathbb{N}}$ is said to satisfy the Cuntz relation, if their range projections $s_is_i^*$ are mutually orthogonal.
The following definition makes sense by the construction above, and will be shown below to be independent of the particular Hilbert space and isometries chosen:

Definition 4.15. *For every $n \in \mathbb{N}_{\geq 2}$ let $(s_i)_{1\leq i\leq n}$ be isometries on a Hilbert space $\mathcal{H}$ which satisfy the Cuntz relation. Then we define $\mathcal{O}_n := C^*_{\mathcal{H}}(s_1,\ldots,s_n)$. That is, $\mathcal{O}_n$ is the sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ generated by n partial isometries which satisfy the Cuntz relation.*
We define $\mathcal{O}_\infty$ as the C^*-algebra generated by an infinite sequence of isometries satisfying the Cuntz relation.

We list the following properties of $\mathcal{O}_n$ without proof:

Theorem 4.16. *For every $n \in \mathbb{N}$ the Cuntz algebra $\mathcal{O}_n$ is a separable, unital, simple, purely infinite, nuclear C^*-algebra.*

In addition, the Cuntz algebras, similar to the Toeplitz algebra, have the following universal property :

Proposition 4.17. *Let $n \in \overline{\mathbb{N}}_{\geq 2}$. For every unital C^*-algebra A containing isometries $(t_i)_{1\leq i\leq n}$ satisfying the Cuntz relation, there is a unique unital *-homomorphism $\varphi : \mathcal{O}_n \to A$ such that $\varphi(s_i) = t_i$.*

Observation 4.18. For any isometries $t_1,\ldots,t_n$ satisfying the Cuntz relation, the *-homomorphism from Proposition 4.17 must be an isomorphism by simplicity of $\mathcal{O}_n$, and therefore $C^*(t_1,\ldots,t_n)$ is isomorphic to $\mathcal{O}_n$.

Corollary 4.19. *Let p be a projection in a C^*-algebra A. Then p is properly infinite if and only if $\mathcal{O}_\infty$ embeds unitally into pAp. In particular, a unital C^*-algebra is properly infinite if and only if it unitally contains $\mathcal{O}_\infty$ (up to isomorphism).*

Proof. By Lemma 3.19, p is properly infinite in pAp, which consequently is a purely infinite C^*-algebra by definition. By Lemma 4.5, there is a sequence of isometries in pAp satisfying the Cuntz relation, and by universality $\mathcal{O}_\infty$ embeds in pAp. $\square$

Proposition 4.20. *For every $n \in \bar{\mathbb{N}}$ we have $\mathcal{O}_n \subseteq \mathcal{O}_2$.*

Proof. As $\mathcal{O}_2$ is purely infinite, we see that $\mathcal{O}_\infty$ embeds unitally into $\mathcal{O}_2$. Fix some $n \in \mathbb{N}$. Let s_1, s_2 be the generators of $\mathcal{O}_2$ and $n \in \mathbb{N}$. Mimicking the proof of Lemma 4.5, we define new isometries by setting for all $k = 1,\ldots n$:

$$t_k := s_1^{k-1}s_2 \qquad \text{and} \qquad t_{n+1} := s_1^n\ .$$

It then follows easily, that $t_n^* t_m = \delta_{n,m}$, and

$$\begin{aligned}1 - \sum_{k=1}^{n} t_k t_k^* &= 1 - \sum_{k=1}^{n} s_1^{k-1} s_2 s_2^* s_1^{*k-1} \\ &= 1 - \sum_{k=1}^{n} s_1^{k-1}(1 - s_1 s_1^*) s_1^{*k-1} \\ &= 1 - \sum_{k=1}^{n} s_1^{k-1} s_1^{*k-1} - s_1^k s_1^{*k} \\ &= s_1^n s_1^{*n} = t_{n+1} t_{n+1}^* ,\end{aligned}$$

and therefore $t_1, \ldots, t_{n+1}$ satisfy the Cuntz relation. By universality, $\mathcal{O}_{n+1}$ embeds in $\mathcal{O}_2$. □

We will need one more fact concerning $\mathcal{O}_n$ later on:

Lemma 4.21. *For every $n, m \in \mathbb{N}$, the matrix algebra M_{n^m} embeds unitally into $\mathcal{O}_n$.*

Proof. For every $m \in \mathbb{N}$ we will show that $\mathcal{O}_n$ contains matrix units for M_{n^m}. In order to do so, we define for every $\mu \in \{1, \ldots, n\}^m$:

$$s_\mu := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_m} .$$

As $|\{1, \ldots, n\}^m| = n^m$, we may choose a bijection $\tau\colon \{1, \ldots, n^m\} \to \{1, \ldots, n\}^m$ and set $g_{ij} := s_{\tau(i)} s_{\tau(j)}^*$. We have $s_\mu^* s_\nu = \delta_{\mu,\nu}$ and hence $g_{ij} g_{kl} = 0$ if $j \neq k$ and $g_{ij} g_{kl} = g_{il}$ if $j = k$. Further, by rearranging

$$\sum_{i=1}^{n^m} g_{ii} = \sum_{i=1}^{n^m} s_{\tau(i)} s_{\tau(i)}^* = 1 ,$$

showing that the g_{ij} form a set of matrix units for M_{n^m}, and hence that $\mathcal{O}_n$ contains a subalgebra isomorphic to M_{n^m}. □

The following will an ingredients to the proof of the $\mathcal{O}_2$-embedding theorem 7.24:

Theorem 4.22. *The Cuntz algebra $\mathcal{O}_2$ is self absorbing, that is:*

$$\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2 .$$

4.4.2 On the Cuntz sum

Remark 4.23. Let s be an isometry in a unital C^*-algebra A and consider the *-conjugation V by s, i.e., $a \mapsto s^* a s$. Then it is plain to see that this is a unital linear map, which is completely positive as $M_n(V) := V_n$ is *-conjugation by the isometry $1_n \otimes s$; it follows, that $||V||_{cb} = ||V(1)|| = 1$ (see for example [Rr1, Chapter 6]). Note further that for all $a \in A$ we have

$$||as^*|| \leq ||a|| = ||as^*s|| \leq ||as^*|| .$$

As for all $a \in A$ we have

$$V(a^*a) - V(a)^*V(a) = V(a^*a) - V(a^*ss^*a) = V(a^*(1-ss^*)a) \geq 0\,,$$

where we used positivity of V and that $(1-ss^*)$ is a projection, and hence $a^*(1-ss^*)a$ positive, to see that the last inequality holds. One might paraphrase this by saying that V is supermultiplicative. In particular, given another isometry t and denoting the associated *-conjugation by W we get

$$W(V(a)^*V(a)) - WV(a)^*WV(a) > 0\,.$$

Definition 4.24 (Cuntz sum). *Let $s_1, s_2 \in A$ be isometries in a unital C^*-algebra satisfying the Cuntz relation. Then we define a binary operation, called the Cuntz sum, by*

$$\oplus_{s_1,s_2} : A \times A \to A,\ (a,b) \mapsto s_1 a s_1^* + s_2 b s_2^*.$$

Remark 4.25. Note that for the Cuntz sum we have the following inequality:

$$||x \oplus_{s_1,s_2} x - x|| \leq ||\big[s_1, x\big]|| + ||\big[s_2, x\big]||.$$

Proof. We have $s_1s_1^* + s_2s_2^* = 1$ and hence:

$$\begin{aligned} ||x \oplus_{s_1,s_2} x - x|| &= ||s_1xs_1^* + s_2xs_2^* - x(s_1s_1^* + s_2s_2^*)|| \\ &= ||\big[s_1, x\big]s_1^* + \big[s_2, x\big]s_2^*|| \leq ||\big[s_1, x\big]|| + ||\big[s_2, x\big]||\,. \end{aligned}$$

□

Lemma 4.26 (cf. [KiRr1, Lemma 2.4]). *Let A be a unital C^*-algebra and assume that s, v_1, $v_2 \in A$ are isometries such that v_1, v_2 satisfy the Cuntz relation. Let V denote *-conjugation by s. Then*

$$w_1 := (1 - ss^*) + sv_1s^* \qquad \textit{and} \qquad w_2 := sv_2$$

again satisfy the Cuntz relation and, for all $a \in A$:

(i) $||a \oplus_{w_1,w_2} V(a) - a|| \leq ||\big[v_1, V(a)\big]|| + ||\big[v_2, V(a)\big]|| + 2||\big[a, ss^*\big]||$,

(ii) $||\big[a, ss^*\big]|| = \max\{||V(a^*a) - V(a)^*V(a)||^{1/2}, ||V(aa^*) - V(a)V(a)^*||^{1/2}\}$.

Proof. The element w_2 is immediately seen to be an isometry, for w_1 this follows from

$$w_1^*w_1 = (1 - ss^*) + (1 - ss^*)sv_1s^* + sv_1^*s^*(1 - ss^*) + sv_1^*v_1s^* = (1 - ss^*) + ss^* = 1\,.$$

That they satisfy the $\mathcal{O}_2$-relation is a consequence of

$$w_1w_1^* + w_2w_2^* = (1 - ss^*) + sv_1s^*sv_1^*s^* + sv_2v_2^*s^* = 1\,.$$

Let $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ be the decomposition of a with respect to ss^* (see Remark 2.3). Employing Proposition 2.2 twice, gives $(\circ)$

$$\begin{aligned} &||\big[a, ss^*\big]|| \\ =& \max\{||(1-ss^*)as||\,, ||s^*a(1-ss^*)||\} \\ =& \max\{||(1-ss^*)asv_1^*s^*||\,, ||sv_1s^*a(1-ss^*)||\} \\ =& ||(1-ss^*)asv_1^*s^* + sv_1s^*a(1-ss^*)||\,. \end{aligned}$$

We prove (i):

$$\begin{aligned} ||a \oplus_{w_1,w_2} V(a) - a|| \leq& ||(1-ss^*)a(1-ss^*) + sv_1V(a)v_1^*s^* + sv_2V(a)v_2^*s^* - a|| \\ &+ ||(1-ss^*)asv_1^*s^* + sv_1s^*a(1-ss^*)|| \\ \overset{(\circ)}{=}& ||(1-ss^*)a(1-ss^*) + ss^*ass^* - a|| \\ &+ ||V(a) \oplus_{v_1,v_2} V(a) - V(a)|| + ||\big[a, ss^*\big]|| \\ \leq& 2||\big[a, ss^*\big]|| + ||\big[v_1, V(a)\big]|| + ||\big[v_2, V(a)\big]||\,, \end{aligned}$$

where we use Remark 4.25 to see the last inequality.
Ad (ii): Proposition 2.2 shows that

$$||a|| = \max\{||ss^*a(1-ss^*)||\,, ||(1-ss^*)ass^*||\}\,.$$

Using

$$\begin{aligned} &||V(a^*a) - V(a)^*V(a)|| = ||s^*a^*as - s^*a^*ss^*as|| \\ =& ||(1-ss^*)as||^2 = ||(1-ss^*)ass^*||^2 \end{aligned}$$

and

$$\begin{aligned} &||V(aa^*) - V(a)V(a)^*|| = ||s^*aa^*s - s^*ass^*a^*s|| \\ =& ||(1-ss^*)a^*ss^*||^2 = ||ss^*a(1-ss^*)||^2\,, \end{aligned}$$

(ii) follows easily. $\square$

5 Real rank for C^*-algebras

In this section we will define the real rank of a C^*-algebra A. It is shown in [BrPed, Proposition 1.1] that for a compact Hausdorff space X the real rank of $C(X)$ and the covering dimension of X are the same, and hence the concept of real rank for C^*-algebras may be regarded as noncommutative dimension theory. The name real rank stems from the fact that the covering dimension of a compact Hausdorff space may be characterized by properties of continuous mappings from the topological space into $\mathbb{R}^n$.

Definition 5.1 (Real rank). *A unital C^*-algebra A has real rank n if for all $k \leq n+1$ and $(a_1, \ldots, a_k) \in A_{sa}^k$ there is for every $\varepsilon > 0$ some $(b_1, \ldots, b_k) \in A_{sa}^k$ with the property that $\sum\limits_{i=1}^{k} b_i^2$ is invertible and*

$$||\sum_{i=1}^{k}(a_i - b_i)^2|| < \varepsilon\,.$$

We will denote the real rank of a unital C^*-algebra A by $\mathrm{RR}(A)$ *and for a non unital C^*-algebra we set* $\mathrm{RR}(A) := \mathrm{RR}(\tilde{A})$.

Even though we have defined real rank in full generality we will be concerned only with the case of real rank zero from now on. In the sequel we will deduce numerous conditions on a C^*-algebra that are equivalent to having real rank zero.

Lemma 5.2 (cf. [BrPed, Lemma 2.3]). *Let A be unital, p a projection in A and a an element in A with decomposition $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ with respect to p (cf. Remark 2.3). If a_4 is invertible in $(1-p)A(1-p)$, then a is invertible in A if and only if $a_1 - a_2 a_4^{-1} a_3$ is invertible in pAp.*

Proof. Observe that

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} p & a_2 a_4^{-1} \\ 0 & (1-p) \end{pmatrix} \begin{pmatrix} a_1 - a_2 a_4^{-1} a_3 & 0 \\ 0 & a_4 \end{pmatrix} \begin{pmatrix} p & 0 \\ a_4^{-1} a_3 & 1-p \end{pmatrix}\,.$$

Now

$$\begin{pmatrix} p & a_2 a_4^{-1} \\ 0 & (1-p) \end{pmatrix} \begin{pmatrix} p & a_2 a_4^{-1} \\ p & (1-p) \end{pmatrix} = \begin{pmatrix} p + a_2 a_4^{-1} p & p a_2 a_4^{-1} + a_2 a_4^{-1}(1-p) \\ 0 & (1-p) \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & (1-p) \end{pmatrix}$$

and invertibility of the right hand factor follows similarly. Therefore a is invertible if and only if

$$\begin{pmatrix} a_1 - a_2 a_4^{-1} a_3 & 0 \\ 0 & a_4 \end{pmatrix}$$

is invertible, which is equivalent to invertibility of $a_1 - a_2 a_4^{-1} a_3$, a_4 being invertible by hypothesis. □

Recall that for every element a of a unital C^*-algebra A with $||1-a|| < 1$ we have invertibility, as $\big(\sum_{n=0}^{N}(1-a)^n\big)_{N\in\mathbb{N}}$ is convergent to a^{-1}, whereas

$$||a^{-1}|| \leq \lim_{N\to\infty}\sum_{n=0}^{N}||1-a||^n = (1-||1-a||)^{-1} \ .$$

Theorem 5.3 (cf. [BrPed, Theorem 2.5]). *Let A be a unital C^*-algebra with real rank zero. Then for every projection p in A the unital C^*-algebras pAp and $(1-p)A(1-p)$ also have real rank zero.*

Proof. Let $a \in (pAp)_{sa}$, $1 > \varepsilon > 0$ and $b \in A_{sa}$ invertible such that $||a+(1-p)-b|| < \varepsilon$. Decompose b with respect to p into b_1, b_2, b_3 and b_4; then b_1 and b_4 are self adjoint, as b was, $b_2 = b_3^*$, and

$$||(1-p)-b_4|| = ||(1-p)a(1-p)+(1-p)-(1-p)b(1-p)|| \leq \varepsilon$$

implying invertibility of b_4 in $(1-p)A(1-p)$. This entails by Lemma 5.2 invertibility of

$$c := b_1 - b_2 b_4^{-1} b_3 \in pAp$$

because b was invertible by hypothesis, and c is self adjoint by the above. As $(1-p)$ is a unit for $(1-p)A(1-p)$, we have, applying the approximation preceding the theorem inside $(1-p)A(1-p)$

$$||b_4^{-1}|| \leq (1-||b_4-(1-p)||)^{-1} \leq (1-\varepsilon)^{-1},$$

and therefore

$$||a-c|| = ||a-b_1||+||b_2 b_4^{-1} b_3|| \leq ||pap-pbp||+(1-\varepsilon)^{-1}\ ||pb(1-p)||^2 \leq \varepsilon+(1-\varepsilon)^{-1}\varepsilon^2 \ ,$$

as was to be shown. □

Lemma 5.4. *Let a and b be self adjoint elements in A and $\varepsilon > 0$ such that*

$$a-\varepsilon \leq b \leq a+\varepsilon \ .$$

Then $||a-b|| \leq \varepsilon$.

Proof. We show that $\sigma(a-b) \subseteq [-\varepsilon, \varepsilon]$, implying

$$||a-b|| = \rho(a-b) \le \varepsilon$$

by [M, Theorem 2.1.1]. To this end, let $\lambda \in \sigma(a-b)$. By hypothesis,

$$\lambda - \varepsilon \in \sigma(a-b-\varepsilon) \subseteq \mathbb{R}^- \qquad \text{and} \qquad \lambda + \varepsilon \in \sigma(a-b+\varepsilon) \subseteq \mathbb{R}^+,$$

implying $\lambda \le \varepsilon$ and $\lambda \ge -\varepsilon$. □

We cite the following Lemma:

Lemma 5.5 (cf. [BrPed, Lemma 2.2]). *Let a and b be self adjoint elements of A and assume $||a-b|| \le \varepsilon$. Then*

$$||a_+ - b_+|| \le \frac{1}{2}\Big(\big((||a|| + ||b||)\varepsilon\big)^{1/2} + \varepsilon\Big).$$

Lemma 5.6. *Let A be a C^*-algebra. Assume that the following condition holds:*
For all positive, orthogonal elements a and b in A and for every $\varepsilon > 0$ there is a projection $p \in A$ such that $||(1-p)a|| < \varepsilon$ and $||pb|| < \varepsilon$.
Then for all $t \in \mathbb{R}$ the set $G_t := \{a \in A_{sa} \mid t \notin \sigma(a)\}$ is a dense open subset of A_{sa}.

Proof. Let $t \in \mathbb{R}$; as the set of invertible elements is open in A, we see that G_t is open. Furthermore let $a \in A_{sa}$ and $\varepsilon > 0$. Then $(a-t)_+$ is orthogonal to $(a-t)_-$, thence there is a projection p in A such that $||(a-t)_+ p|| < \varepsilon$ and $||(a-t)_-(1-p)|| < \varepsilon$. This entails

$$\begin{aligned} ||a - pap - (1-p)a(1-p)|| &= ||(1-p)ap + pa(1-p)|| = ||(1-p)ap|| \\ &= ||(1-p)\big((a-t)_+ - (a-t)_-\big)p|| < 2\varepsilon \end{aligned}$$

by Proposition 2.2. Further we have

$$(t-\varepsilon)p \le tp + p((a-t)_+ - \varepsilon)p \le tp + p\big((a-t)_+ - (a-t)_-\big)p = pap\,,$$

and similarly $(1-p)a(1-p) \le (t+\varepsilon)(1-p)$. This shows that

$$pap + 2\varepsilon p \ge (t+\varepsilon)p \qquad \text{and} \qquad (1-p)a(1-p) - 2\varepsilon(1-p) \le (t-\varepsilon)(1-p)\,,$$

whence the element

$$b := (pap + 2\varepsilon p) + ((1-p)a(1-p) - 2\varepsilon(1-p))$$

is in G_t, as otherwise

$$t \in \sigma_{pAp}(pap + 2\varepsilon p) \;\text{ or } t \in \sigma_{(1-p)A(1-p)}((1-p)a(1-p) - 2\varepsilon(1-p))$$

would have to hold, which is impossible by the above inequalities. Moreover

$$||a-b|| \le ||a - pap - (1-p)a(1-p)|| + ||2\varepsilon p|| + ||2\varepsilon(1-p)|| \le 4\varepsilon\,.$$

□

Proposition 5.7 (cf. [BrPed, Theorem 2.6]). *The following conditions on a C^*-algebra A are equivalent:*

(i) *A has real rank zero*

(ii) *For all positive, orthogonal elements a and b in $\tilde{A}$ and for every $\varepsilon > 0$ there is a projection $p \in \tilde{A}$ such that $||(1-p)a|| < \varepsilon$ and $||pb|| < \varepsilon$.*

(iii) *For all positive elements a and b in $\tilde{A}^+$ and for every $\varepsilon > 0$ with $||ab|| < \varepsilon^2$ there is a projection p in A such that $||(1-p)a|| < \varepsilon$ and $||pb|| < \varepsilon$.*

Proof. We may assume that A is unital.
(i)⇒(iii): The idea is the following: We use the hypothesis on A to find an invertible self adjoint element c which is close to $a-b$. Then we choose p from Proposition 3.10 such that $pc = c_+$ and $(1-p)c = c_-$ and show that it does the job.
Let $1 > \varepsilon > 0$. To begin with, note that $\big((2||a-b|| + \varepsilon_1)\varepsilon_1\big) + \varepsilon_1 \to 0$ for $\varepsilon_1 \to 0$, hence we may choose $\varepsilon_1 > 0$ with

$$||ab||^{1/2} - \frac{1}{2}\Big(\big((2||a-b|| + \varepsilon_1)\varepsilon_1\big)^{1/2} + \varepsilon_1\Big) < \varepsilon\,.$$

Select an invertible self adjoint element c in A such that $||a-b-c|| < \varepsilon_1$. We have, using Lemma 5.5 at $(*)$:

$$\begin{aligned}||a-c_+|| \leq& ||a-(a-b)_+|| + ||(a-b)_+ - c_+|| \overset{*}{\leq} ||ab||^{1/2} + \frac{1}{2}\big((||a-b|| + ||c||)\varepsilon_1 + \varepsilon_1\big)\\ \leq& ||ab||^{1/2} = \frac{1}{2}\Big(\big(||a-b|| + ||c-a-b||)\varepsilon_1\big)^{1/2} + \varepsilon_1\Big) \leq \varepsilon\,.\end{aligned}$$

Now let p be the projection in A with $pc = c_+$ and $(1-p)c = c_-$, then

$$\begin{aligned}||(1-p)a|| =& ||(1-p)a + (1-p)c_+||\\ =& ||1-p||\;||a-c_+|| < \varepsilon\,,\end{aligned}$$

replacing $(1-p)$ with p, which is in turn orthogonal to c_-, the which we may substitute for c_+, allows us to apply the above argument to b instead of a, obtaining $||pb|| < \varepsilon$.
Finally, (iii) obviously implies (ii), and that (ii) implies (i) follows from Lemma 5.6, as we see that G_0, i. e. , the self adjoint and invertible elements, are dense in $\tilde{A}_{sa}$, which is exactly the definition of real rank zero. □

Theorem 5.8 (cf. [Ped4, Proposition 14], [BrPed, Theorem 2.6]). *The following conditions on a C^*-algebra A are equivalent:*

(i) *A has real rank zero,*

(ii) *The self adjoint elements of A with finite spectra are dense in A_{sa},*

(iii) *Every hereditary C^*-subalgebra of A has an approximate unit consisting of projections.*

Proof. To show that (i) implies (ii), observe that if A has real rank zero, than we may apply Lemma 5.6 by Proposition 5.7. Let $a \in A_{sa}$ and $\varepsilon > 0$. Observe that A_{sa} is a complete metric space in the metric induced by the norm on A, whence by Baire's theorem the set $G := \bigcap_{t \in \varepsilon\mathbb{Z}} G_t$ is dense in A_{sa}. We may therefore choose an element $b \in G$ with $||a - b|| < \varepsilon$. Now construct an element close to b with finite spectrum as follows: Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := n\varepsilon$ for $t \in [n\varepsilon, (n+1)\varepsilon)$ and note that f is continuous on $\sigma(b)$. As a consequence

$$||f(b) - b|| = \sup\{|f(t) - t| \mid t \in \sigma(b)\} \leq \varepsilon\,,$$

yielding $||a - f(b)|| \leq 2\varepsilon$. As $\sigma(b)$ is bounded, $f(b)$ has finite spectrum, as desired.
To see that (iii) implies (i), we show that condition (ii) in Proposition 5.7 holds. To this end, let $a, b \in A_+$ and $\varepsilon > 0$. As there is an approximate unit of projections in the hereditary C^*-algebra generated by a, we may choose a projection p in $\overline{aAa}$ with $||pa - a|| = ||(1 - p)a|| < \varepsilon$. As $bc = 0$ for all elements in the hereditary C^*-algebra generated by a, even $bp = 0$ holds.
We refer the reader to [Ped4, Proposition 14] for a proof of the implication (ii)$\Rightarrow$(iii). $\square$

The proof of the following theorem is due to Mikael Rørdam:

Theorem 5.9. *Let A be unital, simple and purely infinite. Then A has real rank zero.*

Proof. Let $a \in A_{sa}$; if a is invertible, then there is nothing to show. Hence we assume that $0 \in \sigma(a)$ and $a \neq 0$. By Proposition 3.10 it further suffices to show that $(a - \varepsilon)_+ \in \overline{A_{sa} \cap Gl(A)}$ holds for all $\varepsilon > 0$. As $(a - \varepsilon)_+$ corresponds to a function being zero on a neighborhood of zero or containing zero as an isolated point, we may choose an element b in the C^*-algebra generated by $(a - \varepsilon)_+$ and 1 such that $||b|| = 1$ and $(a - \varepsilon)_+ b = b(a - \varepsilon)_+ = 0$. By Proposition 4.13 (ii) there is an infinite projection $p \in \overline{bAb}$; observe that $p(a - \varepsilon)_+ = (a - \varepsilon)_+ p = 0$.
Now, as A is simple, p is full, therefore we may choose a projection p_0 with $1 - p \sim p_0 \leq p$, and take a partial isometry $v \in A$ such that $v^*v = 1 - p$ and $vv^* = p_0$. For every $\delta > 0$ we define $c_\delta := (a - \varepsilon)_+ + \delta(p - p_0) + \delta(v + v^*)$. Then c_δ is arbitrarily close to $(a - \varepsilon)_+$ and we show that it is invertible. Using the standard decomposition, this time with respect to $(1 - p)$, p_0 and $p - p_0$, c_δ corresponds to the matrix

$$\begin{pmatrix} (a - \varepsilon)_+ & \delta v^* & 0 \\ \delta v & 0 & 0 \\ 0 & 0 & \delta(p - p_0) \end{pmatrix}.$$

Now $\delta(p - p_0)$ is a scalar multiple of the unit in $(p - p_0)A(p - p_0)$, and it therefore suffices to show that the upper corner is invertible in $(1 - p)A(1 - p) + p_0Ap_0$, but this follows from

$$\begin{pmatrix} a & \delta v^* \\ \delta v & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta^{-1}v^* \\ \delta^{-1}v & -\delta^{-2}vav^* \end{pmatrix} = \begin{pmatrix} v^*v & \delta^{-1}av^* - \delta^{-1}v^*vav^* \\ 0 & vv^* \end{pmatrix} = \begin{pmatrix} (1 - p) & 0 \\ 0 & p_0 \end{pmatrix}.$$

$\square$

6 Discrete crossed products

In this section we will study crossed products of C^*-algebras with discrete groups. We start out with some background on representation theory.

6.1 Covariant representations

Definition 6.1. *Let Γ be a group and $\mathcal{H}$ a Hilbert space; denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. Then a group homomorphism $U\colon \Gamma \to \mathcal{U}(\mathcal{H})$, $\gamma \mapsto U_\gamma$ which is continuous with respect to the strong operator topology is called a unitary representation of Γ on $\mathcal{H}$.*

Definition 6.2. *A triple (A, Γ, α) where A is a C^*-algebra, Γ a locally compact group and $\alpha\colon \Gamma \to \mathrm{Aut}(A)$, $\gamma \mapsto \alpha_\gamma$ a group homomorphism which is continuous with respect to the topology of pointwise convergence on $\mathrm{Aut}(A)$, is a called a (C^*-)dynamical system. We call such a system discrete, if the group Γ is discrete, and countable, given that Γ is countable.*

Recall that continuity with respect to the topology of pointwise convergence on $\mathrm{Aut}(A)$ in the definition above means, that for every $a \in A$ the map $\gamma \mapsto \alpha_\gamma(a)$ is continuous. We will be interested in unitary representations that "tie together" the representations of a group acting on a C^*-algebra and the C^*-algebra itself. Such representations will be used in order to define a norm on the crossed product to be constructed in the next subsection.

Definition 6.3. *A covariant representation of a C^*-dynamical system is a pair (π, U), where $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a *-representation of A and $U\colon \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation of Γ on the same space such that for all $\gamma \in \Gamma$ and $a \in A$ we have:*

$$U_\gamma \pi(a) U_\gamma^* = \pi(\alpha_\gamma(a))\,.$$

Observe that for a discrete group Γ and a C^*-algebra A the set of functions $f\colon \Gamma \to A$ with compact support may be identified with the set $A\Gamma$ of formal finite sums of the form $\sum_{\gamma\in\Gamma} a_\gamma \gamma$, where only finitely many a_γ are nonzero. In the sequel we will use this identification without further comment.

Proposition 6.4. *Let (A, Γ, α) be a discrete dynamical system. We define the following operations for all $a = \sum_{\gamma\in\Gamma} a_\gamma \gamma$, $b = \sum_{\beta\in\Gamma} b_\beta \beta$ in $A\Gamma$:*

(i) $a + b := \sum\limits_{\gamma\in\Gamma} (a_\gamma + b_\gamma)\gamma$

(ii) $(a * b) := \sum_{\gamma\in\Gamma} \big(\sum_{\beta\in\Gamma} a_\gamma \alpha_\gamma(b_{\gamma^{-1}\beta})\big)\beta$

(iii) $(\lambda a) := \sum_{\gamma\in\Gamma} (\lambda a_\gamma)\gamma$ *for all* $\lambda \in \mathbb{C}$

(iv) $a^* := \sum_{\gamma\in\Gamma} \alpha_\gamma(a^*_{\gamma^{-1}})\gamma$.

Then $A\Gamma$ *is a *-algebra with these operations.*

We write $a * b$ as in the case of a non-discrete group this is the convolution of two functions. However, we will use the convention to omit the $*$ when there is no ambiguity.

Proof. Instead of checking on the axioms of a C^*-algebra we use Proposition 6.6 below to show that there is a faithful *-representation of $A\Gamma$ on a Hilbert space $\mathcal{H}$; this will imply that the axioms hold in $A\Gamma$. □

Recall that for every Hilbert space $\mathcal{H}$ and a discrete group Γ we may form the vector space $l^2(\Gamma,\mathcal{H})$ of all square summable functions on Γ with values in $\mathcal{H}$. We endow it with a scalar product by setting for all $x, y \in l^2(\Gamma,\mathcal{H})$

$$\langle x|y\rangle := \sum_{\gamma\in\Gamma} \langle x_\gamma | y_\gamma\rangle\,,$$

thus defining a Hilbert space. The following proposition will prove the existence of certain covariant representations of (A,Γ,α), the so called regular representations.

Proposition 6.5. *Let* (A,Γ,α) *be a discrete dynamical system. For every representation of* $\pi\colon A \to \mathcal{B}(\mathcal{H})$ *of* A *there is a covariant representation* $(\hat\pi, U)$ *of* (A,Γ,α) *on the Hilbert space* $l^2(\Gamma,\mathcal{H})$ *given by*

$$\begin{aligned} \big(U_\gamma(x)\big)(\beta) &:= x(\gamma^{-1}\beta) && \text{for all } x \in l^2(\Gamma,\mathcal{H}),\ \gamma,\beta\in\Gamma,\\ \big(\hat\pi(a)(x)\big)(\beta) &:= \pi(\alpha_{\beta^{-1}}(a))(x(\beta)) && \text{for all } a\in A,\ x\in l^2(\Gamma,\mathcal{H}) \text{ and } \beta\in\Gamma. \end{aligned}$$

In addition, if π *is faithful, so is* $\hat\pi$.

The representation U above is usually called the left regular representation of Γ. It is clear that U is a faithful group representation, i.e., injective.

Proof. First we show that U is a unitary representation of Γ. As Γ is discrete, we do not have to worry about continuity. Further for any $x, y \in l^2(\Gamma,\mathcal{H})$ and $\gamma\in\Gamma$ we have

$$\langle U_\gamma x \mid y\rangle = \sum_{\beta\in\Gamma} \langle x(\gamma^{-1}\beta) \mid y(\beta)\rangle = \sum_{\beta\in\Gamma}\langle x(\beta) \mid y(\gamma\beta)\rangle = \langle x \mid U_{\gamma^{-1}} y\rangle$$

and therefore U_γ is adjointable with $U_\gamma^* = U_{\gamma^{-1}} = U_\gamma^{-1}$, implying continuity at the same time.

Now we show that $\hat{\pi}$ is a representation of A on $l^2(\Gamma,\mathcal{H})$. To this end, let $a,b\in A$, $x\in l^2(\Gamma,\mathcal{H})$ and $\gamma\in\Gamma$. That $\hat{\pi}$ is a linear map follows directly from the definitions, we only show multiplicativity and that $\hat{\pi}$ commutes with $*$. This follows from

$$\begin{aligned}\big(\hat{\pi}(ab)(x)\big)(\gamma) =&\pi\big(\alpha_{\gamma^{-1}}(ab)(x(\gamma)\big)\\ =&\pi(\alpha_{\gamma^{-1}}(a))\circ\pi(\alpha_{\gamma^{-1}}(b))(x(\gamma))\\ =&\pi(\alpha_{\gamma^{-1}}(a))\big((\hat{\pi}(b)(x))(\gamma)\big)\\ =&\big(\hat{\pi}(a)\hat{\pi}(b)(x)\big)(\gamma)\end{aligned}$$

and

$$\begin{aligned}\big(\hat{\pi}(a^*)(x)\big)(\gamma) &= \pi(\alpha_{\gamma^{-1}}(a^*))(x(\gamma))\\ &= \big(\pi(\alpha_{\gamma^{-1}}(a))\big)^*(x(\gamma))\\ &= \big(\hat{\pi}(a)^*(x)\big)(\gamma)\,.\end{aligned}$$

It now only remains to check on covariance of $(\hat{\pi},U)$. For all x, a and γ as above we calculate

$$\begin{aligned}\big(U_\gamma^*\hat{\pi}(a)U_\gamma(x)\big)(\beta) &= \big(\hat{\pi}(a)U_\gamma(x)\big)(\gamma^{-1}\beta)\\ &= \pi(\alpha_{(\gamma^{-1}\beta)^{-1}}(a))\big(U_\gamma(x)\big)(\gamma^{-1}\beta)\\ &= \pi(\alpha_{\beta^{-1}\gamma}(a))\big(x(\gamma\gamma^{-1}\beta)\big)\\ &= \big(\hat{\pi}(\alpha_\gamma(a))(x)\big)(\beta)\,,\end{aligned}$$

which completes the proof. As $\mathcal{H}$ embeds into $l^2(\Gamma,\mathcal{H})$ by sending $\xi\in\mathcal{H}$ to the square summable function on Γ which is nonzero only on e and sends e to ξ, we see that $\hat{\pi}$ is faithful when π is. □

Proposition 6.6. *Let (A,Γ,α) be a dynamical system with a covariant representation (π,U). Then*

$$\rho\Big(\sum_{\gamma\in\Gamma}a_\gamma\gamma\Big) := \sum_{\gamma\in\Gamma}\pi(a_\gamma)U_\gamma \tag{6.1}$$

*defines a *-representation of $A\Gamma$, the which we will denote by $\pi\rtimes U$.*
If (π,U) is the covariant representation constructed in Proposition 6.5 with respect to a faithful representation of A, then $\pi\rtimes U$ is a faithful representation of $A\Gamma$.

The representation $\pi\rtimes U$ obtained from (π,U) is usually called the integrated representation or integrated form. This stems from the fact that in the more general case of a locally compact group Γ we have to use the Haar measure and integrate π and U over Γ in 6.1 in order to obtain a *-representation of $C_c(\Gamma,A)$, which replaces $A\Gamma$ in this case.

Proof. It follows immediately from the definitions that ρ is a linear map. Let $a = \sum_{\gamma\in\Gamma} a_\gamma\gamma$ and $b = \sum_{\beta\in\Gamma} b_\beta\beta$ be elements of $A\Gamma$. Then we get

$$\begin{aligned}\rho(ab) =& \rho\big(\sum_{\gamma,\beta\in\Gamma} a_\gamma\alpha_\gamma(b_{\gamma^{-1}\beta})\beta\big)\\ =& \sum_{\gamma,\beta\in\Gamma} \pi(a_\gamma)\pi(\alpha_\gamma(b_{\gamma^{-1}\beta}))U_\beta\\ =& \sum_{\gamma,\beta\in\Gamma} \pi(a_\gamma)U_\gamma\pi(b_{\gamma^{-1}\beta})U_\gamma^*U_\beta\\ =& \sum_{\gamma,\beta\in\Gamma} \pi(a_\gamma)U_\gamma\pi(b_\beta)U_\gamma^*U_{\gamma\beta}\\ =& \rho(a)\rho(b)\,.\end{aligned}$$

Concerning the involution we have

$$\rho(a)^* = \sum_{\gamma\in\Gamma} U_\gamma^*\pi(a_\gamma)^*U_\gamma U_\gamma^* = \sum_{\gamma\in\Gamma} \pi(\alpha_{\gamma^{-1}}(a_\gamma))U_{\gamma^{-1}} = \rho(a^*)\,,$$

as desired.
Now let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a faithful representation and $(\hat{\pi}, U)$ the regular covariant representation of (A, Γ, α) on $l^2(\Gamma, \mathcal{H})$ as defined in 6.5. Let $0 \neq a = \sum_{\gamma\in\Gamma} a_\gamma\gamma \in A\Gamma$; then there is some $\gamma \in \Gamma$ with $a_\gamma \neq 0$. Choose $\xi \in \mathcal{H}$ such that $\pi(a_\gamma)\xi \neq 0$. Define

$$f\colon \Gamma \to \mathcal{H},\ \begin{cases} f(\nu) = \xi \text{ for } \nu = e\\ f(\nu) = 0 \text{ for } \nu \neq e\end{cases}.$$

Now calculate

$$\begin{aligned}\big(\hat{\pi} \rtimes U(a)\big)(f)(\gamma) &= \big(\sum_{\mu\in\Gamma} \hat{\pi}(a_\mu)U_\mu\big)(f)(\gamma) = \sum_{\mu\in\Gamma} \hat{\pi}(a_\mu)(f)(\mu^{-1}\gamma)\\ &= \sum_{\mu\in\Gamma} \pi(\alpha_{\gamma^{-1}\mu}(a_\mu))(f(\mu^{-1}\gamma)) = \pi(\alpha_{\gamma^{-1}\gamma}(a_\gamma))f(e) = \pi(a_\gamma)(\xi) \neq 0\,.\end{aligned}$$

□

Observation 6.7. Observe that for any dynamical system (A, Γ, α) we have a *-morphism $\iota\colon A \to A\Gamma,\ a \mapsto ae$, called the canonical inclusion. A *-representation $\rho\colon A\Gamma \to \mathcal{B}(\mathcal{H})$ of $A\Gamma$ therefore yields a *-representation $\pi := \rho\,\iota$ of A on $\mathcal{H}$. For every $a \in A$ we thus get $||\rho(ae)|| = ||\pi(a)|| \leq ||a||$, implying for every $\gamma \in \Gamma$

$$||\rho(a\gamma)||^2 = ||\rho(a\gamma)^*\rho(a\gamma)|| = ||\rho(a^*ae)|| = ||a||^2\,.$$

For every *-representation of $A\Gamma$ and $\sum_{\gamma\in\Gamma} a_\gamma\gamma \in A\Gamma$ this entails

$$||\rho(\sum_{\gamma\in\Gamma} a_\gamma\gamma)|| \leq \sum_{\gamma\in\Gamma} ||a_\gamma||\,.$$

Moreover we see that for a net $(a_\lambda)_\lambda$ in A which is convergent to $a \in A$, the net $\big(\rho(a_\lambda\gamma)\big)_\lambda$ converges to $\rho(a\gamma)$ for every $\gamma \in \Gamma$.

Proposition 6.8. *Let (A, Γ, α) be a discrete dynamical system, $\rho : A\Gamma \to \mathcal{B}(\mathcal{H})$ a *-representation and $(u_\lambda)_\lambda$ an approximate unit for A. Then setting*

$$\rho|_A \colon A \to \mathcal{B}(\mathcal{H}),\ a \mapsto \rho(ae)$$
$$\rho|_\Gamma \colon \Gamma \to \mathcal{B}(\mathcal{H}),\ \gamma \mapsto \lim_\lambda \rho(u_\lambda \gamma)\,,$$

where the limit is taken with respect to the strong operator topology (see the proof for convergence), defines a covariant representation $(\rho|_A, \rho|_\Gamma)$ of (A, Γ, α), which is independent of the choice of $(u_\lambda)_\lambda$.

Observe that the process of mapping a given covariant representation (π, U) to its integrated form $\pi \rtimes U$ and then "restricting" as above gives back the representation we started with.

Proof. That $\rho|_A$ is a *-representation of A is contained in Observation 6.7. A simple calculation, using the scalar product on $\mathcal{H}$, shows that the definition of $\rho|_\Gamma$ is independent of the choice of the approximate unit. We proceed to show that the strong limit $\lim_\lambda \rho(u_\lambda\gamma)$ exists for every $\gamma \in \Gamma$. We denote by $A\Gamma\mathcal{H}$ the linear span of elements of the form $\rho(a)\xi$ with $a \in A\Gamma$ and $\xi \in \mathcal{H}$ and by $[A\Gamma\mathcal{H}]$ the closure of this subspace. It suffices to show that the limit exists pointwise on $A\Gamma$ and is bounded; therefore it is enough to show it exists on elements of the form $\rho(a\gamma)\xi$ with $a \in A$, $\gamma \in \Gamma$ and $\xi \in \mathcal{H}$, and is bounded on these. Let $\xi' = \rho(a\mu)\xi$ be such an element. Then

$$\rho(u_\lambda\gamma)\xi' = \rho(u_\lambda\gamma)\rho(a\mu)\xi = \rho(u_\lambda\alpha_\gamma(a)\gamma\mu)\xi\,,$$

and the latter converges by Observation 6.7, hence so does the first. As for all $\xi \in A\Gamma\mathcal{H}$

$$||\rho|_\Gamma(\gamma)\xi|| \leq \lim_\lambda ||\rho(u_\lambda\gamma)||\ ||\xi|| \leq ||\xi||\,,$$

we may extend $\rho|_\Gamma(\gamma)$ by continuity to an operator on all of $[A\Gamma\mathcal{H}]$ and set it as the identity elsewhere. Using that the definition of $\rho|_\Gamma$ does not depend on the particular choice of $(u_\lambda)_\lambda$, a simple calculation shows that $\rho|_\Gamma$ is in fact a group homomorphism $\rho|_\Gamma$ indeed a unitary representation [1] by a straight forward calculation, showing that

$$\langle \rho|_\Gamma(\gamma^{-1})\xi \mid \xi' \rangle = \langle \xi \mid \rho|_\Gamma(\gamma)\xi' \rangle$$

for all $\xi, \xi' \in A\Gamma\mathcal{H}$. It remains to check on covariance of $(\rho|_A, \rho|_\Gamma)$. Let $a \in A$ and $\gamma \in \Gamma$. Then, as multiplication by a fixed operator is strongly continuous

$$\begin{aligned}
\rho|_\Gamma(\gamma)\rho|_A(a)\rho|_\Gamma(\gamma)^* &= \lim_\lambda \rho(u_\lambda\gamma)\rho(ae)\rho|_\Gamma(\gamma)^* \\
&= \lim_\lambda \rho(a_\lambda\alpha_\gamma(a)\gamma) \lim_\lambda \rho|_\Gamma(\gamma)^* \\
&= \lim_\lambda(\alpha_\gamma(a)u_\lambda\gamma\gamma^{-1}) \\
&= \rho|_A(\alpha_\gamma(a))\,.
\end{aligned}$$

[1] Recall that taking adjoints is not strongly continuous and therefore

$$\big(\lim_\lambda \rho(u_\lambda\gamma)\big)^* = \lim_\lambda \rho(u_\lambda\gamma)^* = \lim_\lambda \rho\big((u_\lambda\gamma)^*\big)$$

does not hold in general; thus a calculation is unavoidable.

□

6.2 Discrete crossed products

Definition 6.9. *Let (A, Γ, α) be a discrete C^*-dynamical system. Then we define the crossed product $A \rtimes_\alpha \Gamma$ to be the completion of $A\Gamma$ in the C^*-norm given by*

$$||a|| := \sup_{\rho \text{ *-repr. of } A\Gamma} ||\rho(a)|| \text{ for all } a \in A\Gamma.$$

The reduced crossed product $A \rtimes_{\alpha r} \Gamma$ is the completion in the norm formed by taking the supremum above only over the regular representations.

By means of Proposition 6.6 we have a faithful regular representation if we use the integrated form obtained from the universal representation of A, and therefore $||a||_{A \rtimes_\alpha \Gamma} \neq 0$ for every nonzero a in $A\Gamma$. Further we obtain from Observation 6.7 that the supremum lies in $\mathbb{R}$ and hence we have, in fact, defined a norm.
Note that for a dynamical system (A, Γ, α), where Γ is countable and A separable, the crossed product is again separable. This follows, as the set of finitely supported functions from Γ into S, where S is a countable dense subset of A, is again countable, and dense in AG, hence in $A \rtimes_\alpha \Gamma$ by Observation 6.7.

Proposition 6.10. *Let (A, Γ, α) be a discrete dynamical system. Then the crossed product $A \rtimes_\alpha \Gamma$ has the following property: For every covariant representation (π, U) of (A, Γ, α) over $\mathcal{H}$ there is a *-representation ρ of $A \rtimes_\alpha \Gamma$ on $\mathcal{H}$ such that $\rho\,\iota = \pi$, where ι is the canonical inclusion of A.*

Proof. By Proposition 6.1, the integrated form of (π, U) is a *-representation of $A\Gamma$, which we may extend to $A \rtimes_\alpha \Gamma$, as it is a bounded linear map on $A\Gamma$ by construction of the norm on the crossed product. The definition of the integrated form yields

$$\pi \rtimes U\,\iota(a) = \pi \rtimes U(ae) = \pi(a)U(e) = \pi(a)\,.$$

as was to be shown. □

Observation 6.11. Note that we may use the embedding $a \mapsto ae$ already encountered in Observation 6.7 to see that A embeds isometrically into $A \rtimes_\alpha \Gamma$. If A is unital, then we further have an embedding $\kappa\colon \Gamma \to A \rtimes_\alpha \Gamma$ by setting $\kappa(\gamma) := 1_A\gamma$. As there is a faithful representation of $A \rtimes_\alpha \Gamma$ by the GNS-construction, and as A embeds into $A \rtimes_\alpha \Gamma$, we may as well from now on fix a Hilbert space $\mathcal{H}$ such that $A, A \rtimes_\alpha \Gamma \subseteq \mathcal{B}(\mathcal{H})$. If A is unital, then we also assume $\Gamma \subseteq \mathcal{B}(\mathcal{H})$.

Lemma 6.12. *Let I be an ideal in a C^*-algebra A and $\pi : I \to \mathcal{B}(\mathcal{H})$ a non-degenerate representation of I on a Hilbert space $\mathcal{H}$. Then there exists a canonical extension $\hat{\pi} : A \to \mathcal{B}(\mathcal{H})$ of π to A.*

Proof. We essentially pursue the same idea as in Proposition 6.8. Let $(u_\lambda)_\lambda$ be an approximate unit of I. For every $a \in A$ and $\xi' \in \mathcal{H}$ such that $\xi' = \pi(b)\xi$ for some $b \in I$ and $\xi \in \mathcal{H}$, we get

$$\pi(au_\lambda)\xi' = \pi(au_\lambda)\pi(b)\xi = \pi(au_\lambda b)\xi$$

and therefore $\lim_\lambda \pi(au_\lambda)\xi$ exists on a dense subset of $\mathcal{H}$, as π is non degenerate; we further have

$$||\hat{\pi}(a)\xi'|| = ||\lim_\lambda \pi(au_\lambda)\xi|| \leq ||a||\, ||\xi'||$$

showing that we may extend $\hat{\pi}(a)$ to all of $\mathcal{H}$. Easy calculations show that $\hat{\pi}$ is a *-representation of A, which extends π by

$$\hat{\pi}(a) = \lim_\lambda \pi(au_\lambda) = \pi(a)$$

for all $a \in I$, because π is norm decreasing. □

Proposition 6.13. *Let (A, Γ, α) be a discrete dynamical system. Let I be an ideal in A such that $\alpha_\gamma(I) \subseteq I$ for all $\gamma \in \Gamma$ (a Γ-invariant ideal). Then*

$$\overline{I\Gamma}^{||\cdot||_{A\rtimes_\alpha\Gamma}} \cong I \rtimes_{\alpha|_I} \Gamma \,,$$

that is, we may identify the crossed product of the dynamical system $(I, \Gamma, \alpha|_I)$ with the ideal in $A \rtimes_\alpha \Gamma$ generated by $I\Gamma$.

Proof. Let $\iota : I\Gamma \to A \rtimes_\alpha \Gamma$ be the inclusion map. Then this is clearly a *-morphism, and it remains to show that ι extends isometrically to $I \rtimes_{\alpha|_I} \Gamma$. As every *-representation of $A\Gamma$ yields a *-representation of $I\Gamma$ by restriction, ι is norm decreasing and extends to a *-homomorphism on the full crossed product $I \rtimes_\alpha \Gamma$. Given a non-degenerate *-representation ρ of $I\Gamma$, we obtain a covariant representation $(\rho|_I, \rho|_\Gamma)$ by Proposition 6.8. Use Lemma 6.12 to extend $\rho|_I$ to a representation $\hat{\rho}|_I$ of A and manufacture a covariant representation $(\hat{\rho}|_I, \rho|_\Gamma)$ of (A, Γ, α). Tracing back the definitions of the representations involved, we see that

$$\rho|_\Gamma(\gamma)\hat{\rho}|_I(a)\rho|_\Gamma(\gamma)^* = \rho|_\Gamma \lim_\lambda \rho|_I(u_\lambda a)\rho|_\Gamma(\gamma)^* = \lim_\lambda \rho|_I(\alpha_\gamma(u_\lambda a)) = \hat{\rho}|_I(\alpha_\gamma(a))\,.$$

Therefore the integrated form $\hat{\rho}|_I \rtimes \rho|_\Gamma$ of $(\hat{\rho}|_I, \rho|_\Gamma)$ gives a *-representation of $A\Gamma$, and consequently ι is an isometric embedding of $I \rtimes_{\alpha|_I} \Gamma$ into $A \rtimes_\alpha \Gamma$. □

Example 6.14. Let τ be a *-automorphism of A. Then we define an action α of $\mathbb{Z}$ on A by

$$\alpha_n(a) := \tau^n(a) \text{ for all } n \in \mathbb{Z} \text{ and } a \in A.$$

In this situation, we write $A \rtimes_\alpha \mathbb{Z} = A \rtimes_\tau \mathbb{Z}$. Consequently, the formula $U_1 a U_1^* = \tau(a)$ implies that there is a unitary, namely U_1, in $\mathcal{B}(\mathcal{H})$ such that $U_1^n a U_1^{n*} = \tau^n(a)$. The unitary U_1 is called the unitary implementing (the action of) τ. By construction,

$$\{ \sum_{n=-N}^{N} a_n u^n \mid N \in \mathbb{N},\ a_n \in A\}$$

is then a dense subset of $A \rtimes_\tau \mathbb{Z}$. Further, by [M][Theorem 3.1.8] it follows that u is an element of the multiplier algebra $\mathcal{M}(A)$ of A.
In case that A is not unital, we may extend τ to an automorphism $\tilde{\tau}$ of $\tilde{A}$ by functoriality of $\tilde{\cdot}$ and use the above argument on $\tilde{A}$, and by Proposition 6.13 we may then view $A \rtimes_\tau \mathbb{Z}$ as an ideal to again obtain a unitary as above.
The universality for crossed products translates in this special case to:

Proposition 6.15. *Let A be a unital C^*-algebra, τ an automorphism of A implemented by a unitary u. If B is another unital C^*-algebra such that there is a *-homomorphism $\varphi : A \to B$ and a unitary $v \in B$ such that $\varphi(\tau(a)) = v^*\varphi(a)v$, then there is a *-homomorphism $\bar{\varphi} : A \rtimes_\tau \mathbb{Z} \to B$ such that $\bar{\varphi}|_A = \varphi$ and $\bar{\varphi}(u) = v$, that is, we have a commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, "\tau"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \arrow[dr, hook, "\iota"] & \\
& & A \rtimes \mathbb{Z} \arrow[dl, dotted, "\bar{\varphi}"] \\
B \arrow[r, "Ad(v)"] & B &
\end{tikzcd}$$

6.2.1 The circle action

Lemma 6.16. *Let $(A, \mathbb{Z}, \tau)$ be a C^*-algebra and u the unitary implementing the action of τ. For every $t \in \mathbb{T}$ denote by $\hat{\alpha}_t$ the automorphism of $A \rtimes_\tau \mathbb{Z}$ defined by extending*

$$\hat{\alpha}_t(\sum_{n\in\mathbb{Z}} a_n u^n) := \sum_{n\in\mathbb{Z}} a_n (tu)^n \text{ (finite sums).}$$

Then $(A \rtimes_\tau \mathbb{Z}, \mathbb{T}, \hat{\alpha})$ is a C^*-dynamical system.

Proof. Let $a = \sum_{n\in\mathbb{Z}} a_n u^n, b = \sum_{n\in\mathbb{Z}} b_n u^n \in A\mathbb{Z}$. Then for every $t \in \mathbb{T}$

$$\begin{aligned}
\hat{\alpha}_t(a+b) &= (\sum_{n\in\mathbb{Z}} (a_n + b_n)(tu)^n) = (\sum_{n\in\mathbb{Z}} a_n (tu)^n) + (\sum_{n\in\mathbb{Z}} b_n (tu)^n) = \hat{\alpha}_t(a) + \hat{\alpha}_t(b)\,, \\
\hat{\alpha}_t(ab) &= \hat{\alpha}_t(\sum_{n,m\in\mathbb{Z}} a_n b_m u^{n+m}) = \sum_{n,m\in\mathbb{Z}} a_n b_m (tu)^n (tu)^m = \hat{\alpha}_t(a)\hat{\alpha}_t(b)\,, \\
\hat{\alpha}_t(a^*) &= \hat{\alpha}_t(\sum_{n\in\mathbb{Z}} u^{*n} a_n^*) = \hat{\alpha}_t(\sum_{n\in\mathbb{Z}} \tau^{-n}(a_n^*) u^{*n}) = \sum_{n\in\mathbb{Z}} \tau^{-n}(a_n^*)(tu)^{*n} \\
&= \sum_{n\in\mathbb{Z}} \tau^{-n}(a_n^*) \bar{t}^n u^{*n} = \sum_{n\in\mathbb{Z}} \bar{t}^n u^{*n}(a_n^*) = \hat{\alpha}_t(a)^*\,.
\end{aligned}$$

As $\hat{\alpha}_t$ has clearly norm one, it extends to a *-isomorphism on $A \rtimes_\tau \mathbb{Z}$. A simple calculation shows, that $\hat{\alpha} : \mathbb{T} \to A \rtimes_\tau \mathbb{Z}$ is a group homomorphism. It remains to show that $\hat{\alpha}$ is continuous with respect to the topology of pointwise convergence. If $t_k \to t$ for $k \to \infty$ in $\mathbb{T}$, then we have

$$||\hat{\alpha}_{t_k}(a) - \hat{\alpha}_t(a)|| \leq \sum_{n \in \mathbb{Z}} ||a_n t_k^n u^n - a_n t^n u^n|| = \sum_{n \in \mathbb{Z}} |t_k^n - t^n| \, ||a_n u^n|| \to 0\,.$$

□

Definition 6.17. *Let B be a sub-C^*-algebra of a C^*-algebra A. An expectation of A onto B is a linear positive map $E : A \to B$, such that $E^2(a) = E(a)$ for all $a \in A$. The expectation E is called faithful, if, whenever a is a positive element in A, then $E(a) = 0$ if and only if $a = 0$.*

We quote the following Proposition from [Dav] without proof, where the integral is the usual Banach-space-valued one:

Proposition 6.18 (cf. [Dav, Theorem VIII.2.1]). *Let $(A, \mathbb{Z}, \tau)$ be a C^*-dynamical system and $(A \rtimes_\tau \mathbb{Z}, \mathbb{T}, \hat{\alpha})$ be as in Lemma 6.16. Then*

$$E\colon A \rtimes_\tau \mathbb{Z} \to A,\ a \mapsto \int_{t \in \mathbb{T}} \hat{\alpha}_t(a) dt$$

defines a faithful conditional expectation, called the canonical expectation onto A.

Theorem 6.19. *Let $(A \rtimes_\tau \mathbb{Z}, \mathbb{T}, \hat{\alpha})$ be as in Lemma 6.16 and u the unitary implementing the action of τ. Let B be a C^*-algebra, $\varphi\colon A \to B$ a *-homomorphism, v a unitary in B such that $\varphi(\tau(a)) = v\varphi(a)v^*$, and denote by $\bar{\varphi}$ the induced morphism (see Proposition 6.15). If there is a group homomorphism $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ such that $(B, \mathbb{T}, \beta)$ is a C^*-dynamical system and the following diagram commutes for all $t \in \mathbb{T}$*

$$\begin{tikzcd}
A \rtimes_\tau \mathbb{Z} \arrow[r, "\bar{\varphi}"] \arrow[d, "\hat{\alpha}_t"'] & B \arrow[d, "\beta_t"] \\
A \rtimes_\tau \mathbb{Z} \arrow[r, "\bar{\varphi}"'] & B
\end{tikzcd}$$

then $\bar{\varphi}$ is injective if and only if φ is.

Note that in the category of C^*-dynamical systems, this is just existence of an induced morphism (i.e., a morphism coming from an equivariant *-homomorphism) between the systems $(A \rtimes_\tau \mathbb{Z}, \mathbb{T}, \hat{\alpha})$ and $(B, \mathbb{T}, \beta)$.

Proof. Set

$$F\colon B \to B,\ b \mapsto \int_{t \in \mathbb{T}} \beta_t(b)\, \mathrm{d}\, t\,.$$

For all $a \in A \rtimes_\tau \mathbb{Z}$ this definition entails

$$F\,\bar{\varphi}(a) = \int_{t\in\mathbb{T}} \alpha_t(\bar{\varphi}(a))\,\mathrm{d}\,t = \int_{t\in\mathbb{T}} \bar{\varphi}(\hat{\alpha}_t(a))\,\mathrm{d}\,t \overset{(*)}{=} \bar{\varphi}\int_{t\in\mathbb{T}} \hat{\alpha}_t(a)\,\mathrm{d}\,t = \bar{\varphi}\,E(a)\,,$$

where $(*)$ follows from the fact that $\bar{\varphi}$ is a *-homomorphism and from the definition of the integral. Note that $\bar{\varphi}|_A = \varphi$.
In other "words", we have commutativity of

$$\begin{tikzcd} A \rtimes_\tau \mathbb{Z} \arrow[r, "\bar{\varphi}"] \arrow[d, "E"'] & B \arrow[d, "F"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

Now let $a \in \mathrm{Ker}(\bar{\varphi})$. Then by multiplicativity of $\bar{\varphi}$ we get $a^*a \in \mathrm{Ker}(\bar{\varphi})$, hence

$$0 = F\,\bar{\varphi}(a^*a) = \bar{\varphi}\,E(a^*a) = \varphi\,E(a^*a)\,,$$

and if φ is injective, then $E(a^*a) = 0$. But E is faithful, hence $a^*a = 0$, and by the C^*-equation, $a = 0$ follows. As the restriction of $\bar{\varphi}$ to A is φ, it is clearly a necessary condition that φ is injective. □

The next Lemma will be an ingredient of the embedding theorem Theorem 7.24. Roughly speaking, it forces an injective *-homomorphism from a crossed product by $\mathbb{Z}$ into an arbitrary algebra to be injective, by tensoring the latter one with $C(\mathbb{T})$.

Lemma 6.20. *Let $A \rtimes_\tau \mathbb{Z}$ be a crossed product, where τ is an automorphism of A induced by a unitary u, further $\varphi\colon A \to B$ a *-homomorphism such that there is a unitary $v \in B$ with $\varphi(\tau(a)) = v\varphi(a)v^*$ for all $a \in A$. Denote the canonical circle action on $A \rtimes_\tau \mathbb{Z}$ again by $\hat{\alpha}$. If z is the canonical generator of $C(\mathbb{T})$, then*

$$\psi\colon A \rtimes_\tau \mathbb{Z} \to B \otimes C(\mathbb{T}),\ \sum_{n\in\mathbb{Z}} a_n u^n \mapsto \sum_{n\in\mathbb{Z}} \varphi(a_n) v^n \otimes z^n$$

*defines a *-homomorphism, which is injective if and only if φ is.*

Proof. First of all, ψ is induced by the map φ' obtained by composing φ with the embedding $d \mapsto d \otimes 1$ of B into $B \otimes C(\mathbb{T})$, which satisfies

$$\varphi'(\tau(a)) = \varphi(\tau(a)) \otimes 1 = v^n\varphi(a)v^{*n} \otimes z^n z^{*n} = v^n \otimes z^n(\varphi'(a))(v^n \otimes z^n)^*\,.$$

Next, note that $\mathbb{T}$ acts naturally on $C(\mathbb{T})$, namely by rotation $\gamma_t(z) := tz$, where z is the canonical generator. Define $\beta_t := id_B \otimes \gamma_t$ for all $t \in C(\mathbb{T})$, to obtain an action of $\mathbb{T}$ on $B \otimes C(\mathbb{T})$. Furthermore, letting $a = \sum_{n\in\mathbb{Z}} a_n u^n \in A\mathbb{Z}$

$$\beta_t\psi(a) = \sum_{n\in\mathbb{Z}} \varphi(a_n)v^n \otimes t^n z^n = \sum_{n\in\mathbb{Z}} t^n\varphi(a_n)v^n \otimes z^n = \psi\hat{\alpha}_t(a)\,.$$

The claim is now a consequence of Theorem 6.19. □

We have the following proposition concerning tensor products and crossed products:

Proposition 6.21. *Let A and B be C^*-algebras and τ an automorphism of A. Then*

$$(A \otimes_{max} B) \rtimes_{\alpha \otimes \mathrm{id}_B} \mathbb{Z} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes_{max} B\,.$$

Proof. Let u be the unitary implementing the action of α. The map $\varphi : A \otimes_{max} B \to (A \rtimes_\alpha \mathbb{Z}) \otimes_{max} B$ induced by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ has the property required for an application of the universal property, namely

$$\varphi(\tau \otimes \mathrm{id}_B(a \otimes b)) = \varphi(uau^* \otimes b) = uau^* \otimes b = (u \otimes 1)(a \otimes b)(u \otimes 1)^*$$

Denote by $\bar{\varphi}$ the induced map. As there is a canonical circle action on $A \rtimes_\tau \mathbb{Z} \otimes_{\max} B$ given by $\hat{\alpha}_t \otimes \mathrm{id}_B$ for all $t \in \mathbb{T}$ which commutes with $\bar{\varphi}$, we see that $\bar{\varphi}$ is injective by Theorem 6.19 □

The projection mentioned in the next proposition is called the Rieffel projection.

Proposition 6.22. *Let τ be the automorphism of $C_0(\mathbb{R})$ given by $\tau(f)(t) := f(1+t)$ for all $f \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$. Then there is a nonzero projection p in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$.*

Proof. We define functions $f, g \in C_0(\mathbb{R})$ by $f(t) := 1 - |t|$ for $t \in [-1, 1]$ and $f(t) := 0$ elsewhere. Further set $g(t) := \sqrt{f(t) - f(t)^2}$ for $t \in [0, 1]$ and $g(t) := 0$ elsewhere. Then we have for all $t \in [-1, 0]$

$$\begin{aligned}\tau(g)(t)^2 = g(1+t)^2 &= f(1+t) - f(1+t)^2 = 1 - |1+t| - (1 - |1+t|)^2 \\ &= -(t + t^2) = 1 - |t| - (1 - |t|)^2 = f(t) - f(t)^2 = g(t)^2\end{aligned}$$

implying

$$\Big(\tau(g^2) + f^2 + g^2\Big)(t) = \begin{cases} f(t) & \text{for } 0 \le t \le 1 \\ \tau(g)(t)^2 - f^2(t) = g(t)^2 - f^2(t) = f(t) & \text{for } -1 \le t \le 0 \end{cases}.$$

Hence $\tau(g^2) + f^2 + g^2 = f$. Let u be the unitary implementing the action and set $p := ug + f + gu^*$; we show that p is a projection. Note first, that by an easy calculation using the definition of f and g we get $(ug)^2 = 0$ and $ug(f + \tau^{-1}(f)) = ug$. As p is clearly self adjoint, it follows that p is a projection from

$$\begin{aligned}p^2 &= (ug)^2 + ugf + uggu^* + fug + f^2 + fgu^* + gu^*ug + gu^*f + gu^*gu^* \\ &= ug(f + \tau^{-1}(f)) + \tau(g^2) + f^2 + g^2 + (f + \tau^{-1}(f))gu^* \\ &= ug + f + gu^*\,.\end{aligned}$$

□

Proposition 6.23. *Let τ be the automorphism of $C_0(\mathbb{R})$ defined in 6.22. Then there is an embedding of A in $C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z}$.*

Proof. By Proposition 6.21 we have

$$C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z} \cong \big(C_0(\mathbb{R}) \otimes A\big) \rtimes_{\tau \otimes \mathrm{id}_A} \mathbb{Z} \cong (C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A,$$

hence we may use the embedding $a \mapsto p \otimes a$, where p is the projection from Proposition 6.22. □

7 Kirchberg algebras

Definition 7.1. *A Kirchberg algebra is a separable, simple, nuclear, and purely infinite C^*-algebra.*

7.1 Stinespring revisited

In this section we give a characterization similar to the Stinespring characterization for completely positive maps in the special case of a unital, completely positive, nuclear map ρ from a unital Kirchberg algebra A to itself. This will be achieved by first proving some properties (Proposition 7.4 and 7.8) for certain maps from A into $\mathbb{C}$ and from M_n into A, and then applying these to the factorization, by nuclearity, of the map ρ (see 7.12).
The reader has probably seen the following preparatory lemma as part of the Gel'fand-Naimark-Segal construction.

Lemma 7.2. *Let $\mu\colon A \to \mathbb{C}$ be a positive linear functional on a C^*-algebra. Then the left kernel $\mathcal{N}_l(\mu) := \{a \in A \mid \mu(a^*a) = 0\}$ is a left ideal in A.*

Proof. Let $a, b \in \mathcal{N}_l(\mu)$ and $z \in A$. Then $(x, y) \mapsto \mu(x^*y)$ is a positive sesquilinear form (as $x^*x \geq 0$ for all $x \in A$ and μ is positive) and hence we may apply the Cauchy-Schwarz inequality, giving

$$|\mu(a^*b)| \leq \mu(a^*a)^{1/2}\mu(b^*b)^{1/2} = 0,$$

which implies that $\mathcal{N}_l(\mu)$ is closed under addition by

$$\mu((a+b)^*(a+b)) = \mu(a^*a) + \mu(b^*b) + \mu(a^*b) + \mu(b^*a) = 0\,.$$

We deduce from $z^*z \leq ||z^*z|| = ||z||^2$ (for example by functional calculus) that $a^*z^*za \leq ||z||^2a^*a$. As a consequence we have

$$0 \leq \mu((za)^*za) \leq \mu(||z||^2a^*a) = ||z||^2\mu(a^*a) = 0$$

which implies $za \in \mathcal{N}_l(\mu)$, and so $\mathcal{N}_l(\mu)$ is a left ideal in A. □

Remark 7.3. Let $\omega \in \mathcal{S}(A)$ (the space of states on A). Then we have $\omega(1) = 1$. This follows immediately as 1 constitutes a constant approximate unit and therefore $\omega(1) = ||\omega|| = 1$ (see [M, Theorem 3.3.3]). In the converse direction, we know that a unital positive map $\omega\colon A \to \mathbb{C}$ is a state, as $||\omega|| = \omega(1) = 1$ again by [M, Theorem 3.3.3]. Hence a positive map on a unital C^*-algebra is a state iff it is unital.

Proposition 7.4 (cf. [Rr1, Proposition 6.3.1]). *Let A be a unital, simple, purely infinite C^*-algebra and $\omega\colon A \to \mathbb{C}$ a state on A. Let $F \subseteq A$ be finite and $\varepsilon \geq 0$. Then there is a nonzero projection $p \in A$ with $||pap - \omega(a)p|| \leq \varepsilon$ for all $a \in F$.*

The idea is to approximate ω by a net $(\mu_x)_{x\in X}$ of pure states on F, and for each of these to construct a net $(p_y)_{y\in Y}$ of projections with $\lim_y ||p_y a p_y - \mu_x(a)p_y|| = 0$ on F; having done so we approximate ω by a pure state and the according net of projections.

Proof. The set of pure states is weak*-dense in the states on A by [Dix, Lemme 11.2.4], and so one finds a net $(\mu_x)_{x\in X}$ of pure states which is weak*-convergent to ω on F. Choose $x_0 \in X$ with $||\mu_{x_0}(a) - \omega(a)|| \leq \frac{\varepsilon}{2}$ for all $a \in F$.
Now set $N := \mathcal{N}_l(\mu_{x_0}) \cap \mathcal{N}_l(\mu_{x_0})^*$, thus obtaining an hereditary C^*-subalgebra of A by Lemma 7.2 and [M, Theorem 3.2.1]. By Theorem 5.9, A has real rank zero, so there exists an approximate unit $(q_y)_{y\in Y}$ of projections (see Theorem 5.8) in N. Then we have $\mu_{x_0}(q_y) = \mu_{x_0}(q_y^* q_y) = 0$ for all $y \in Y$ and setting $p_y := 1 - q_y$ we obtain a net of projections with $\mu_{x_0}(p_y) = \mu_{x_0}(1) - \mu_{x_0}(q_y) = \mu_{x_0}(1) = 1$ by Remark 7.3, and therefore $p_y \neq 0$ for all $y \in Y$ (as the states give rise to representations via the GNS-construction and hence, the universal representation being faithful, an element $a \in A$ is zero if a^*a maps to zero under all states; alternatively use [M, Theorem 3.3.6]).
We will show that $\lim_y ||p_y a p_y - \mu_{x_0}(a)p_y|| = 0$ for all $a \in A$. To this end, let $c \in \mathcal{N}_l(\mu_{x_0})$. Then we obtain $c^*c \in N$, as $\mathcal{N}_l(\mu_{x_o})$ is a left ideal, and

$$||cp_y||^2 = ||(cp_y)^* cp_y|| = ||p_y^* c^* c p_y|| \leq ||p_y||\,||c^* c p_y|| \tag{7.1}$$

$$= ||c^*c(1-q_y)|| = ||c^*c - c^*cq_y|| \longrightarrow 0 \tag{7.2}$$

for $(q_y)_{y\in Y}$ was chosen as an approximate unit. If $c^* \in \mathcal{N}_l(\mu_{x_0})$, we get $cc^* \in N$ and $p_y c \to 0$.
Let $a \in A$. We then have $\mathcal{N}(\mu_{x_0}) := \ker(\mu_{x_0}) = \mathcal{N}_l(\mu_{x_0}) + \mathcal{N}_l(\mu_{x_0})^*$ by [Ped2, Proposition 3.13.6] for the pure state μ_{x_0}, and as $a - \mu_{x_0}(a)\cdot 1_A \in \mathcal{N}(\mu_{x_0})$ we may choose $u, v \in \mathcal{N}_l(\mu_{x_0})$ with $a - \mu_{x_0}(a)\cdot 1_A = u + v^*$. Hence we have

$$p_y a p_y - \mu_{x_0}(a)p_y = p_y(a - \mu_{x_0}(a))p_y = p_y(u + v^*)p_y = p_y(up_y + (p_y v)^*) \longrightarrow 0\,.$$

Now choose $y_0 \in Y$ with $||p_{y_0} a p_{y_0} - \mu_{x_0}(a)p_{y_0}|| \leq \frac{\varepsilon}{2}$ for all $a \in A$.
We deduce for all $a \in F$ that

$$\begin{aligned}||p_{y_0} a p_{y_0} - \omega(a)p_{y_0}|| &= ||p_{y_0} a p_{y_0} - \mu_{x_0}(a)p_{y_0} + \mu_{x_0}(a)p_{y_0} - \omega(a)p_{y_0}||\\ &\leq ||p_{y_0} a p_{y_0} - \mu_{x_0}(a)p_{y_0}|| + ||\mu_{x_0}(a) - \omega(a)||\,||p_{y_0}|| \leq \varepsilon,\end{aligned}$$

which is the assertion. □

Observation 7.5. The element $p_n := \frac{1}{\sqrt{n}} \sum_{i,j=1}^{n} e_{ij} \otimes e_{ij} \in M_n \otimes M_n$ is a projection for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Then p_n is obviously self adjoint, and

$$\begin{aligned}(\sqrt{n} \cdot p_n)^2 &= \sum_{i,j,k,l=1}^{n} (e_{ij} \otimes e_{ij})(e_{kl} \otimes e_{kl}) = \sum_{i,j,k,l=1}^{n} e_{ij}e_{kl} \otimes e_{ij}e_{kl} \\ &= \sum_{i,j,l=1}^{n} e_{ij}e_{jl} \otimes e_{ij}e_{jl} = \sum_{i,j,l=1}^{n} e_{il} \otimes e_{il} = n \cdot p_n \,,\end{aligned}$$

as claimed. □

Lemma 7.6. *Let A be a unital C^*-algebra and $n \in \mathbb{N}$. Furthermore let $\eta \colon M_n \to A$ be a completely positive map. Then there exists an element $v \in M_n \otimes M_n \otimes A$ such that $e_{11} \otimes e_{11} \otimes \eta(x) = v^*(x \otimes 1_n \otimes 1_A)v$.*
In particular, if η is unital:

$$v^*v = v^*(1_n \otimes 1_n \otimes 1_A)v = e_{11} \otimes e_{11} \otimes \eta(1_n) = e_{11} \otimes e_{11} \otimes 1_A.$$

Proof. Let $\{e_{ij} \mid i,j = 1,\dots,n\}$ be a system of matrix units in M_n. By definition the map $id_{M_n} \otimes \eta$ is positive and hence

$$y := (id_{M_n} \otimes \eta)(p_n) = \sum_{i,j=1}^{n} e_{ij} \otimes \eta(e_{ij}) \in (M_n \otimes A)^+$$

by the preceding Lemma. Therefore we may take the square root $y^{1/2}$ of y and find a_{ij}, $i,j = 1,\dots,n$, with $y^{1/2} = \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij}$, as $\{e_{ij} \mid i,j = 1,\dots,n\}$ constitutes a basis for M_n. It follows that

$$\begin{aligned}\sum_{i,j=1}^{n} e_{ij} \otimes \eta(e_{ij}) &= y = y^{1/2}y^{1/2} = (y^{1/2})^*y^{1/2} = \left(\sum_{k,i=1}^{n} e_{ki} \otimes a_{ki}\right)^* \sum_{l,j=1}^{n} e_{lj} \otimes a_{lj} \\ &= \sum_{i,k,l,j=1}^{n} e_{ik}e_{lj} \otimes a_{ki}^*a_{lj} \overset{k=l}{=} \sum_{i,j,k=1}^{n} e_{ij} \otimes a_{ki}^*a_{kj} = \sum_{i,j=1}^{n} e_{ij} \otimes \left(\sum_{k=1}^{n} a_{ki}^*a_{kj}\right),\end{aligned}$$

implying $(*)$: $\eta(e_{ij}) = \sum_{k=1}^{n} a_{ki}^*a_{kj}$. Now set $v := \sum_{i,j=1}^{n} e_{i1} \otimes e_{j1} \otimes a_{ji}$. Then for all $i,j = 1,\dots,n$

$$\begin{aligned}v^*(e_{ij} \otimes 1_n \otimes 1_A)v &= \left(\sum_{k,l=1}^{n} e_{k1}^*e_{ij} \otimes e_{l1}^* \otimes a_{lk}^*\right) v = \left(\sum_{l=1}^{n} e_{1j} \otimes e_{1l} \otimes a_{li}^*\right) v \\ &= \sum_{l,r,s=1}^{n} e_{1j}e_{r1} \otimes e_{1l}e_{s1} \otimes a_{li}^*a_{sr} \overset{(r=j,s=l)}{=} \sum_{l=1}^{n} e_{11} \otimes e_{11} \otimes a_{li}^*a_{lj} \\ &= e_{11} \otimes e_{11} \otimes \sum_{l=1}^{n} a_{li}^*a_{lj} \overset{(*)}{=} e_{11} \otimes e_{11} \otimes \eta(e_{ij})\,,\end{aligned}$$

and consequently the two linear maps are equal. □

Lemma 7.7. *Let A be a unital and properly infinite C^*-algebra. Then there are a *-homomorphism $\psi\colon M_n \otimes M_n \otimes A \to A$ and an isometry $\tilde{s} \in A$ such that*

$$\psi(e_{11} \otimes e_{11} \otimes a) = \tilde{s} a \tilde{s}^*$$

for all $a \in A$.

Proof. Let $\{g_{ij} \mid i, j = 1, \ldots, n^2\}$ be the standard matrix basis for $M_n \otimes M_n \simeq M_{n^2}$ with $g_{11} = e_{11} \otimes e_{11}$. As A is properly infinite, we may choose isometries $(t_i)_{i=1,\ldots,n^2}$ with orthogonal range projections by Lemma 4.5 and Observation 4.14.
We set $\psi(g_{ij} \otimes a) := t_i a t_j^*$ for all $i, j = 1, \ldots, n^2$. Then this gives a linear map, and further for $i, j, k, l = 1, \ldots, n^2$ and $a, a' \in A$:

$$\begin{aligned} \psi\left((g_{ij} \otimes a)(g_{kl} \otimes a')\right) &= \psi(g_{ij} g_{kl} \otimes aa') \\ = &\begin{Bmatrix} \psi(0_n \otimes 0_n \otimes aa') = 0_A = t_i a t_j^* t_k a' t_l & \text{for } j \neq k \\ \psi(g_{il} \otimes aa') = t_i aa' t_l = t_i a t_j^* t_k a' t_l & \text{for } j = k \end{Bmatrix} \\ &= \psi(g_{ij} \otimes a)\psi(g_{kl} \otimes a')\,, \end{aligned}$$

As

$$(\psi(g_{ij} \otimes a))^* = (t_i a t_j^*)^* = t_j a^* t_i^* = \psi\left(g_{ji} \otimes a^*\right) = \psi\left((g_{ij} \otimes a)^*\right)$$

we see that ψ is a *-homomorphism and hence is continuous. Taking t_1 as $\tilde{s}$ now gives the Lemma. ☐

Proposition 7.8 (cf. [Rr1, Lemma 6.3.2]). *Let A be a unital, simple, purely infinite C^*-algebra, $n \in \mathbb{N}$, and $\eta\colon M_n \to A$ a completely positive map. Then there exists a $*-$homomorphism $\varphi\colon M_n \to A$ and an isometry $s \in A$ with $\eta(x) = s^* \varphi(x) s$ for all $x \in M_n$.*

Proof. Choose v as in Lemma 7.6 and $\psi, \tilde{s}$ as in Lemma 7.7. Note that

$$\tilde{s}\tilde{s}^* = \psi(e_{11} \otimes e_{11} \otimes 1_A) = \psi(v^* v)\,. \tag{7.3}$$

Set $s := \psi(v)\tilde{s}$ and $\varphi := \psi \circ \iota_{1_n \otimes 1_A}(x)$ (see Lemma 2.7).
Then φ is a *-homomorphism as composition of such and s is an isometry by

$$s^* s = \tilde{s}^* \psi(v)\psi(v^*)\tilde{s} = \tilde{s}^* \psi(vv^*)\tilde{s} \overset{7.3}{=} \tilde{s}^* \tilde{s}\tilde{s}^* \tilde{s} = 1\,.$$

It follows for all $x \in M_n$:

$$\begin{aligned} s^* \varphi(x) s =& \tilde{s}^* \psi(v)^* \psi(x \otimes 1_n \otimes 1_A)\psi(v)\tilde{s} = \tilde{s}^* \left(\psi(v^*(x \otimes 1_n \otimes 1_A)v)\right)\tilde{s} \\ =& \tilde{s}^* \left(\psi(e_{11} \otimes e_{11} \otimes \eta(x))\right)\tilde{s} = \tilde{s}^* \tilde{s}\eta(x)\tilde{s}^* \tilde{s} = \eta(x)\,. \end{aligned}$$

☐

The next Lemma gives us the possibility to transform a matrix-valued linear map into a functional in a “sensible” way, i.e., without spoiling positivity; this process will be shown to be reversible.

Lemma 7.9. *Let A be a unital C^*-algebra and $X \subseteq A$ an operator system in A, further $\sigma : X \to M_n$ a bounded linear map. Denote by $\pi_{ij} : M_n \to \mathbb{C}$ the projection onto the $(i,j)-$th component of a matrix and set:*

$$\omega : M_n \otimes X \to \mathbb{C}, \; \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} \mapsto \frac{1}{n} \sum_{i,j=1}^{n} \pi_{ij}\sigma(a_{ij}) \, .$$

Then ω is a bounded linear map with $\sigma(a) = n \sum_{i,j=1}^{n} \omega(e_{ij} \otimes a) e_{ij}$ for all $a \in X$ which is unital if σ was unital and positive if σ was completely positive.

Proof. Let $a, b \in M_n \otimes X$ and $a = \sum\limits_{i,j=1}^{n} e_{ij} \otimes a_{ij}$, $b = \sum\limits_{i,j=1}^{n} e_{ij} \otimes b_{ij}$ where $a_{ij}, b_{ij} \in X$ for all $i, j = 1, \dots, n$. Then:

$$\begin{aligned}\omega(a+b) = \omega\left(\sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} + b_{ij} \right) &= \sum_{i,j=1}^{n} \pi_{ij}\sigma(a_{ij} + b_{ij}) \\ = \sum_{i,j=1}^{n} \pi_{ij}\sigma(a_{ij}) + \sum_{i,j=1}^{n} \pi_{ij}\sigma(b_{ij}) &= \omega(a) + \omega(b) \, ,\end{aligned}$$

and $\omega(\lambda a) = \lambda \omega(a)$ for all $\lambda \in \mathbb{C}$ is obvious. Therefore ω is linear and it is clearly continuous; moreover we have:

$$\sigma(a) = \sum_{i,j=1}^{n} \pi_{ij}(\sigma(a)) e_{ij} = n \sum_{i,j=1}^{n} \omega(e_{ij} \otimes a) e_{ij}$$

for all $a \in A$. Now let $\sigma(1_A) = 1_n$. Then

$$\omega(\sum_{i,j=1}^{n} e_{ij} \otimes 1_A) = \frac{1}{n} \sum_{i,j=1}^{n} \pi_{ij}\sigma(1_A) = 1_n \, .$$

□

A map $\varphi \colon X \to Y$ between normed spaces will be called the point-norm limit of a set Ψ of maps from X to Y if for every finite $F \subseteq X$ and $\varepsilon > 0$ there is a $\psi \in \Psi$ such that $||\psi(x) - \varphi(x)|| \leq \varepsilon$ for all $x \in F$.

Lemma 7.10. *Let A be a unital, simple, purely infinite C^*-algebra, $\sigma \colon A \to M_n$ unital completely positive and $\varphi \colon M_n \to A$ a *-homomorphism. Then $\varphi \circ \sigma$ is the point-norm limit of *- conjugations.*

Proof. Let $\varepsilon \geq 0$ and $F \subseteq A$ finite. Use Lemma 7.9 and Remark 7.3 to produce a state $\omega \colon M_n \otimes A \to \mathbb{C}$ from σ, and find a nonzero projection $p \in M_n \otimes A$ with $||p(e_{ij} \otimes a)p - \omega(e_{ij} \otimes a)p|| < \varepsilon$ on F from Proposition 6.3.1. In the sequel, we will construct an element u from p such that $\varphi \circ \sigma$ is, at least on F, approximately given by *-conjugation with u. As $M_n \otimes A$ is purely infinite, we have $e_{11} \otimes \varphi(e_{11}) \precsim p$ for the projection $e_{11} \otimes \varphi(e_{11})$ (see the section on simple purely infinite C^*-algebras or Definition 4.6), and hence there is a partial isometry $v \in M_n \otimes A$ with $v^*v = e_{11} \otimes \varphi(e_{11})$ and $vv^* \leq p$ by Proposition 3.29. We now show that $v = \sum_{i,j=1}^n e_{j1} \otimes v_j$ for some $v_1, \ldots, v_n \in A$, that is , the matrix corresponding to v has nonzero entries only in the first column. This follows from

$$e_{11} \otimes \varphi(e_{11}) = v^*v(k,k) = \left(\sum_{r=1}^n v_{rm}^* v_{rn}\right)_{m,n} (k,k) = \sum_{r=1}^n v_{rk}^* v_{rk} \geq 0$$

for all $k \neq 0$, implying $\sum\limits_{r=1}^n |v_{rk}|^2 = 0$, and therefore $v_{rk} = 0$ for all $k \neq 1$ and $r = 1, \ldots, n$. Note that

$$v^*v = v^*vv^*v = v^*vv^*pv = v^*pv\,.$$

Using 2.2, it follows for all $i, j = 1, \ldots, n$ and $a \in A$:

$$\begin{aligned}
&||v_i^* a v_j - \omega(e_{ij} \otimes a)\varphi(e_{11})|| \\
=&||e_{11} \otimes v_i^* a v_j - \omega(e_{ij} \otimes a) e_{11} \otimes \varphi(e_{11})|| \\
=&|| \sum_{r,s=1}^n e_{1r} e_{ij} e_{s1} \otimes v_r^* a v_s - \omega(e_{ij} \otimes a) v^*v|| \\
=&||(\sum_{r=1}^n e_{1r} \otimes v_r^*)(e_{ij} \otimes a)(\sum_{s=1}^n e_{s1} \otimes v_s) - \omega(e_{ij} \otimes a) v^*pv|| \\
=&||v^*p(e_{ij} \otimes a)pv - v^*\omega(e_{ij} \otimes a)pv|| \\
\leq&||v^*||\,||p(e_{ij} \otimes a)p - \omega(e_{ij} \otimes a)p||\,||v|| \leq \varepsilon\,.
\end{aligned}$$

Now set $u := \sqrt{n}\sum_{j=1}^n v_j \varphi(e_{1j})$. Then we obtain for all $a \in A$

$$\begin{aligned}
||u^*au - \varphi(\sigma(a))|| &= ||n \sum_{i,j=1}^n \varphi(e_{1i})^* v_i^* a v_j \varphi(e_{1j}) - \varphi\big(n \sum_{i,j=1}^n \omega(e_{ij} \otimes a) e_{ij}\big)|| \\
&= ||n \sum_{i,j=1}^n \varphi(e_{1i}^*)\, v_i^* a v_j \varphi(e_{1j}) - n \sum_{i,j=1}^n \omega(e_{ij} \otimes a) \varphi(e_{ij})|| \\
&= n|| \sum_{i,j=1}^n \varphi(e_{i1}) v_i^* a v_j \varphi(e_{1j}) - \varphi(e_{i1}) \omega(e_{ij} \otimes a) \varphi(e_{11}) \varphi(e_{1j})|| \\
&= n|| \sum_{i,j=1}^n \varphi(e_{i1}) \Big(v_i^* a v_j - \omega(e_{ij} \otimes a) \varphi(e_{11}) \Big) \varphi(e_{1j})|| \\
&\leq n \sum_{i,j=1}^n ||\varphi(e_{i1})||\, ||v_i^* a v_j - \omega(e_{ij} \otimes a)\varphi(e_{11})||\, ||\varphi(e_{1j})|| \leq n^3 \varepsilon\,,
\end{aligned}$$

whereby the proof is complete.

Note, however, that in addition we have the following equality

$$u\,\varphi(1_n) = \big(\sqrt{n}\sum_{i=1}^{n} v_i\varphi(e_{1i})\big)\varphi(1_n) = \sqrt{n}\sum_{i=1}^{n} v_i\varphi(e_{1i}) = u\,.$$

□

Lemma 7.11. *Let $\varphi, \psi : A \to B$ be continuous maps between C^*-algebras, and $X \subseteq A$ bounded. Assume that φ is the point-norm limit of *-conjugations (with elements of X) of ψ, that is, for every finite subset F of A and $\varepsilon > 0$ there is some $x \in X$ with $||x^*\psi(a)x - \varphi(a)|| < \varepsilon$ for all $a \in F$.*
Then if A is separable, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that φ is the pointwise limit of $(x_n^*\psi x_n)_{n\in\mathbb{N}}$.

Proof. Let $F_1, F_2, F_3 \ldots$ be a sequence of finite subsets of A such that their union is dense in A. For every $n \in \mathbb{N}$ choose $x_n \in X$ such that $||x_n^*\psi(a)x_n - \varphi(a)|| < \frac{1}{n}$ for all $a \in F_n$. Then we get for any $a \in \bigcup\limits_{i\in\mathbb{N}} F_i$, choosing $k \in \mathbb{N}$ with $a \in F_k$:

$$\lim_{n\to\infty} ||x_n^*\psi(a)x_n - \varphi(a)|| = \lim_{k\leq n\to\infty} ||x_n^*\varphi(a)x_n - \psi(a)|| \leq \lim_{k\leq n\to\infty} \frac{1}{n} = 0.$$

The following argument, which extends this result to the whole of A, is used frequently; it will be worked out in detail only this time:
Let $a \in A$ and $(a_k)_k$ a sequence of elements converging to a, where $a_k \in \bigcup_{i\in\mathbb{N}} F_i$ for all $k \in \mathbb{N}$. We derive

$$\begin{aligned}\lim_{n\to\infty} ||x_i^*\varphi(a)x_i - \psi(a)|| &= \lim_{n\to\infty} ||x_n^*\psi(\lim_{k\to\infty} a_k)x_n - \varphi(\lim_{k\to\infty} a_k)|| \\ &= \lim_{k\to\infty}\lim_{n\to\infty} ||x_n^*\psi(a_k)x_n - \varphi(a_k)|| = 0.\end{aligned}$$

□

The following proposition gives the characterization mentioned at the beginning of this section and will go into Lemma 7.21, which in turn is an ingredient to the proof of the exact embedding theorem 7.24.

Proposition 7.12 (cf. [KiPhi, Proposition 1.4]; [Rr1, Proposition 6.3.3]). *Let A be unital, simple and purely infinite, and $\rho\colon A \to A$ a unital nuclear completely positive map. Then ρ is the point-norm limit of *-conjugations by isometries.*
*If A is a Kirchberg algebra, then ρ is the pointwise limit of *-conjugations with isometries.*

Proof. Let $F \subseteq A$ be finite and $0 < \varepsilon \leq 1$. By scaling, we may assume without loss of generality that $1_A \in F$ and $||a|| \leq 1$ for all $a \in F$.
Find unital completely positive maps $\sigma\colon A \to M_n$, $\eta\colon M_n \to A$ with

$$||\rho(a) - \eta\,\sigma(a)|| \leq \frac{\varepsilon}{2} \tag{7.4}$$

for all $a \in F$ as in the definition of nuclearity, and find an isometry $s \in A$ and a *-homomorphism $\varphi\colon M_n \to A$ with

$$\eta(x) = s^*\varphi(x)s \tag{7.5}$$

for all $x \in M_n$ by virtue of Lemma 7.8. Next find, by means of Lemma 7.10, an element $u \in A$ with

$$||u^*au - \varphi\,\sigma(a)|| \leq \frac{\varepsilon}{2} \tag{7.6}$$

for all $a \in F$. As σ is unital and $1_A \in F$, we have

$$||\varphi(1_n) - u^*u|| = ||\varphi\,\sigma(1_A) - u^*u|| \leq \frac{\varepsilon}{2}$$

and it will be shown below that we may even assume $u^*u = \varphi(1_n)$. Thence setting $t := us$ yields an isometry by $t^*t = s^*\varphi(1_n)s = 1$ with

$$\begin{aligned} ||\rho(a) - t^*at|| &\leq ||\rho(a) - \eta\,\sigma(a)|| + ||\eta\,\sigma(a) - t^*at|| \\ &\underset{7.5}{\overset{7.4}{\leq}} \frac{\varepsilon}{2} + ||s^*(\varphi\,\sigma)(a)s - t^*at|| \\ &= \frac{\varepsilon}{2} + ||s^*||\,||\varphi\,\sigma(a) - u^*au||\,||s|| \\ &\overset{7.6}{\leq} \varepsilon. \end{aligned}$$

We proceed to show the assumption on u, first checking on the hypothesis of Lemma 2.9. We have

$$||u^*u - \varphi(1)|| \leq \tfrac{\varepsilon}{2} < 1 \qquad \text{and} \qquad u\varphi(1_n) = \varphi(1_n)$$

by definition of u, as noted at the end of the proof of Lemma 7.10. The norm of u is further controlled, as it is close to a projection, by

$$||u||^2 = ||u^*u|| \leq ||u^*u - p|| + ||p|| \leq \varepsilon + 1.$$

Therefore we may apply Lemma 2.9 to the projection $\varphi(1_n)$ and u to get an element $v \in A$ with $v^*v = \varphi(1_n)$ and $||v - u|| \leq ||u^*u - \varphi(1_n)|| < \varepsilon$. Hence

$$\begin{aligned} ||u^*au - v^*av|| &= ||u^*au - u^*av + u^*av - v^*av|| \\ &\leq ||u - v||(||u^*a|| + ||av||) \\ &\leq ||u - v||\,||a||\big(||u|| + ||v||\big) \\ &\leq ||u - v||\,||a||\big(||u|| + ||v - u|| + ||u||\big) \\ &< \varepsilon(2 + 3\varepsilon)\,. \end{aligned}$$

We estimate

$$||v^*av - \varphi\,\sigma(a)|| \leq ||v^*av - u^*au|| + ||u^*au - \varphi\,\sigma(a)|| < \varepsilon + \varepsilon(2 + 3\varepsilon)\,,$$

which proves the first part of the lemma.
The second part follows directly from Lemma 7.11, taking X as the set of isometries in A and ψ as the identity. $\square$

7.2 Extension

The next theorem is a kind of approximate completely positive extension theorem as, boldly speaking, we might identify E with its image under the injective map ρ and then view the new map η as an approximate extension of σ to B_1.

Proposition 7.13 (cf. [KiPhi, Proposition 1.7]; [Rr1, Proposition 6.3.4]). *Let E be a finite dimensional operator system in a unital separable exact C^*-algebra A and $\varepsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that for all unital separable C^*- algebras B_1, B_2 with B_2 nuclear and $\rho\colon E \to B_1$, $\sigma\colon E \to B_2$ unital completely positive maps subject to*

(i) *ρ is injective and,*

(ii) *$||\rho_n^{-1}|| \leq 1 + \varepsilon/2$ with $\rho^{-1}\colon \rho(E) \to E$,*

there is a unital completely positive map $\eta\colon B_1 \to B_2$ with $||\eta\,\rho - \sigma|| < \varepsilon$.

$$\begin{tikzcd}
 & E \arrow[d, tail, "\rho"] \arrow[dr, "\sigma"] & \\
\rho(E) \arrow[ur, "\rho^{-1}"] \arrow[r, hook, "i"] & B_1 \arrow[r, dotted, "\eta"] & B_2
\end{tikzcd}$$

Proof. The proof will be obtained by constructing the following almost commutative diagram:

$$\begin{tikzcd}
 & & E \arrow[r, "id_E"] \arrow[d, dotted, "\sigma_1\, ucp"] \arrow[dll, "\rho\, ucp"'] & E \arrow[d, dotted, "\sigma_2\, ucp"] \arrow[dr, "\sigma\, ucp"] & \\
\rho(E) \arrow[r, hook, "i"'] & B_1 \arrow[r, dotted, "\tau_1\, ucp"'] & M_n \arrow[r, dotted, "\tau_2\, ucp"'] \arrow[ur, dotted, "\eta_1\, ucb"] & M_r \arrow[r, dotted, "\eta_2\, ucp"'] & B_2
\end{tikzcd}$$

and then taking $\eta := \eta_2\,\tau_2\,\tau_1$.
Second triangle: We apply [Rr1, 6.1.12]. Hence there is $n \in \mathbb{N}$ such that we can choose a unital completely positive map $\sigma_1\colon E \to M_n$ and a unital completely bounded map $\eta_1\colon \sigma_1(E) \to E$ as with $||\eta_1||_{cb} \leq 1 + \varepsilon/4$ and $\sigma_1\,\eta_1 = id_E$
Fourth triangle: The map σ is nuclear by [Rr1, 6.1.3] as B_2 is; we may thus take $r \in \mathbb{N}$ and unital completely positive maps $\sigma_2\colon E \to M_r$, $\eta_2\colon M_r \to B_2$ from the definition of nuclearity such that $||\sigma - \eta_2\,\sigma_2|| \leq \varepsilon/4$, using the fact that E is finite dimensional.
Third triangle: Set $F := \sigma_1(E)$ and consider the map $\sigma_2 \circ \eta_1\colon F \to M_r$. We check on the hypothesis of [Rr1, Lemma 6.1.7]: As σ_1 is unital completely positive we have $||\sigma_1||_{cb} = 1$, and obtain

$$\begin{aligned}
||\sigma_2\,\eta_1||_{cb} &= \sup_{n\in\mathbb{N}} ||id_{M_n} \otimes \sigma_2\,\eta_1|| = \sup_{n\in\mathbb{N}} ||(id_{M_n} \otimes \sigma_2)\,(id_{M_n} \otimes \eta_1)|| \\
&\leq \sup_{n\in\mathbb{N}} ||id_{M_n} \otimes \sigma_2|| \sup_{n\in\mathbb{N}} ||id_{M_n} \otimes \eta_1|| = ||\eta_1||_{cb} = 1 + \varepsilon/4\,,
\end{aligned}$$

whereas $\sigma_2\,\eta_1$ is a unital completely bounded map. Extension by [Rr1, Lemma 6.1.7] gives a unital completely positive map $\tau_2\colon M_n \to M_r$ with

$$||\tau_2|_F - \sigma_2\,\eta_1||_{cb} \leq ||\sigma_2\,\eta_1||_{cb} - 1 \leq \varepsilon/4\,.$$

First triangle:
Put $G := \rho(E)$. Then $\sigma_1\,\rho^{-1}$ is a linear map from $G \to M_n$ and hence has, by [Pa, Proposition 7.9], cb-norm equal to

$$||id_{M_n} \otimes \sigma_1\,\rho^{-1}|| = ||(id_{M_n} \otimes \sigma_1)(id_{M_n} \otimes \rho^{-1})|| \leq ||id_{M_n} \otimes \rho^{-1}|| \leq 1 + \varepsilon/2\,.$$

By [Rr1, Lemma 6.1.7] and we may choose an extension $\tau_1\colon B_1 \to M_n$ such that $||\tau_1|_G - \sigma_1\,\rho^{-1}|| \leq \varepsilon/2$. Note that

$$||\tau_1\,\rho - \sigma_1|| \leq ||\tau_1|_G - \sigma\,\rho^{-1}||\,||\rho|| \leq \varepsilon/2\,.$$

Conclusion: To complete the proof, set $\eta := \eta_2\,\tau_2\,\tau_1$ we now obtain

$$\begin{aligned}
||\eta\,\rho - \sigma|| \leq& ||\eta_2\,\tau_2\,\tau_1\,\rho - \eta_2\,\sigma_2|| + ||\eta_2\,\sigma_2 - \sigma||\\
\leq& ||\eta_2||\,||\tau_2\,\tau_1\,\rho - \sigma_2\,\eta_1\,\sigma_1|| + \varepsilon/4\\
\leq& ||(\tau_2\,\tau_1\,\rho - \tau_2\,\sigma_1) + (\tau_2\,\sigma_1 - \sigma_2\,\eta_1\,\sigma_1)|| + \varepsilon/4\\
\leq& ||\tau_2||\,||\tau_1\,\rho - \sigma_1|| + ||\sigma_1||\,||\tau_2|_F - \sigma_2\eta_1|| + \varepsilon/4 \leq \varepsilon,
\end{aligned}$$

as was to be shown. □

Corollary 7.14 (cf. [Rr1, 6.3.5]). *Let A, B_1, B_2 be a unital, separable C^*-algebras, and A exact, B_1 nuclear.*
*(i) Let $\rho\colon A \to B_1$ and $\sigma\colon A \to B_2$ be unital *-homomorphisms and ρ injective. Then there is a sequence $(\eta_n)_{n\in\mathbb{N}}$ of unital completely positive maps $B_1 \to B_2$ such that for all $a \in A$:*

$$(\eta_n\,\rho)(a) \to \sigma(a) \text{ for } n \to \infty.$$

*If in addition $B_1 = B_2$ is a Kirchberg algebra, then we can even choose the maps η_n to be *-conjugations by isometries.*
*(ii) Let w be a free ultrafilter on $\mathbb{N}$ and $(\rho_n)_{n\in\mathbb{N}}$ a sequence of unital completely positive maps $A \to B_1$, $(\sigma_n)_{n\in\mathbb{N}}$ a sequence of unital completely positive maps from $A \to B_2$. Assume further that the map $\rho\colon A \to (B_1)_\omega$, $a \mapsto \pi_\omega(\rho_n(a))$ is an injective *-homomorphism.*
Then there is a sequence of unital completely positive maps $(\eta_n)_{n\in\mathbb{N}}$ such that:

$$\lim_\omega ||(\eta \circ \rho_n(a) - \sigma_n(a)|| = 0$$

for all $a \in A$. If in addition $B_1 = B_2$ is a Kirchberg algebra, then there exists a sequence of isometries $(s_n)_{n\in\mathbb{N}}$ such that for all $a \in A$

$$\lim_\omega ||s_n^*\rho_n(a)s_n - \sigma_n(a)|| = 0.$$

Proof. (i) Recall first that *-homomorphisms are completely positive and injective *-homomorphisms are isometric. Choose an increasing sequence of finite dimensional operator systems $(E_i)_i$ with dense union in A. For every $n \in \mathbb{N}$ find a unital completely positive map η_n as in 7.13 such that $||(\eta_n \circ \rho - \sigma)|_{E_n}|| \leq \frac{1}{n}$. Then for any $a \in \bigcup_{n\in\mathbb{N}} E_n$ there is a $k \in \mathbb{N}$ such that $a \in E_k$ and it follows that

$$\lim_{n\to\infty} ||(\eta_n \circ \rho - \sigma)(a)|| = \lim_{k\leq n\to\infty} ||(\eta_n \circ \rho - \sigma)|_{E_k}(a)|| \leq \lim_{k\leq n\to\infty} \frac{||a||}{n} = 0,$$

as all maps involved are continuous and $\bigcup_{n\in\mathbb{N}} E_n$ is dense in A this holds for all $a \in A$.
Let now $B_1 = B_2$ be a Kirchberg algebra. Then take for every $n \in \mathbb{N}$ an isometry s_n for η_n by virtue of Proposition 7.12 with $||s_n^* a s_n - \eta_n(a)|| < \frac{1}{n}$ to obtain

$$\lim_{n\to\infty} \big(s_n^*\rho(a)s_n - \sigma(a)\big) = \lim_{n\to\infty} \big(s_n^*\rho(a)s_n - \eta_n \circ \rho(a) + \eta_n \circ \rho(a)) - \sigma(a)\big)$$

$$= \lim_{n\to\infty} \big(s_n^*\rho(a)s_n - \eta_n(\rho(a))\big) + \lim_{n\to\infty} \big(\eta_n \circ \rho(a)) - \sigma(a)\big) = 0\,.$$

The proof of (ii) is contained in the proof of Lemma 7.21 below, therefore we omit it here. □

7.3 The $\mathcal{O}_2$-embedding theorem

Lemma 7.15. *Let A be a unital C^*-subalgebra of a unital C^*-algebra D, $s \in D$ an isometry. If the *- conjugation by s is multiplicative then s^*s commutes with all elements in A, and if it is the identity on A then s commutes with all elements in A.*

Proof. Assume that $a \mapsto s^*as$ is multiplicative and set $p := ss^*$. To begin with, let $u \in \mathcal{U}(A)$. We have

$$(pup)^*(pup) = s(s^*u^*s)(s^*us)s^* = s(s^*u^*us)s^* = ss^*ss^* = p.$$

We may without loss of generality suppose that $A \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Let $\xi \in p(\mathcal{H})$ be a unit vector. Then

$$1 = ||u\xi||^2 = ||up\xi||^2 = ||pup\xi + (1-p)up\xi||^2 = ||pup\xi||^2 + ||(1-p)up\xi||^2$$

$$= ||(pup)^*(pup)\xi||^2 + ||(1-p)up\xi||^2 = ||p\xi||^2 + ||(1-p)up\xi||^2 = 1 + ||(1-p)up\xi||^2$$

by Pythagoras and the above equality.
Now we get

$$0 = (1-p)up = up - pup \qquad \text{and} \qquad (1-p)u^*p = u^*p - pu^*p = 0,$$

therefore u and p commute by Observation 2.1.
For an arbitrary element $u \in A$ with $||u|| \leq 1$ we may decompose u into unitaries in A, and these commute with p. Hence, scaling an arbitrary element, we get the first

part of the theorem.
Finally, given $s^*as = a$ for all $a \in A$, then

$$as = aps = pas = ss^*as = sa$$

as claimed. □

We cite the following theorem for further reference:

Theorem 7.16. *Let A be a unital separable C^*-algebra. Then the following conditions are equivalent:*

(i) *A is $\mathcal{O}_2$-absorbing (i.e. $A \simeq A \otimes \mathcal{O}_2$)*

(ii) *There is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of unital *-homomorphisms $\mathcal{O}_2 \to A$ such that for all $a \in A$ and $x \in \mathcal{O}_2$ we have $\lim_{n\to\infty} ||\big[\varphi_n(x), a\big]|| = 0$ (in the sequel called an approximately central sequence).*

Lemma 7.17 (cf. [KiPhi, Lemma 1.12]; [KiRr1, Lemma 2.5]). *Let A be a unital separable $\mathcal{O}_2$-absorbing C^*-algebra, $s, t \in A$ isometries, and denote by V, W respectively the according *-conjugations. Let F be a finite subset of A and set*

$$\kappa := \kappa^{s,t}_{F\cup F^*} := \max_{a\in F\cup F^*}\{||V(a^*a) - V(a)^*V(a)||^{1/2}, ||WV(a^*a) - WV(a)^*WV(a)||^{1/2},$$
$$||WV(a) - a||\}.$$

For every $\varepsilon > 0$ there exists a unitary u in A such that

$$||u^*V(a)u - a|| \leq 5\kappa + \varepsilon$$

for all $a \in F$.

Proof. By Theorem 7.16 there exists an asymptotically central sequence of unital *-homomorphisms $(\varphi_n)_{n\in\mathbb{N}}$. Let $\bar{s}_1$ and $\bar{s}_2$ be the generators of $\mathcal{O}_2$.
Choose $n \in \mathbb{N}$ such that for all $a \in F$:

$$||[\varphi_n(\bar{s}_1), V(a)]|| + ||[\varphi_n(\bar{s}_2), V(a)]|| \leq \frac{\varepsilon}{2}, \tag{7.7}$$

and

$$||[\varphi_n(\bar{s}_1), WV(a)]|| + ||[\varphi_n(\bar{s}_2), WV(a)]|| \leq \frac{\varepsilon}{2}, \tag{7.8}$$

using that $(\varphi_n)_{n\in\mathbb{N}}$ is asymptotically central. Also observe that $r_1 := \varphi_n(\bar{s}_1)$ and $r_2 := \varphi_n(\bar{s}_2)$ are isometries satisfying the $\mathcal{O}_2$-relation (as φ_n is a unital *-homomorphism). Now set

$$s_1 := (1 - ss^*) + s^*r_1s, \qquad s_2 := sr_2, \qquad \text{and} \qquad t_1 := (1 - tt^*) + t^*r_1t, \qquad t_2 := tr_2\,.$$

Then by Lemma 4.26 s_1, s_2 and t_1, t_2 are isometries satisfying the Cuntz relation. Define $u := t_1 s_2^* + t_2 s_1^*$. We will show that u has the desired properties.
To see that u is a unitary is easy, as

$$u^* u = s_2 t_1^* u + s_1 t_2^* u = s_2 s_2^* + s_1 s_1^* = 1, \qquad uu^* = us_2 t_1^* + us_1 t_2^* = t_1 t_1^* + t_2 t_2^* = 1.$$

The rest requires some more work and we will use:

$$u^*(a \oplus_{t_1,t_2} b)u = u^* t_1 a t_1^* u + u^* t_2 a t_2^* u = s_2 a s_2^* + s_1 b s_1^* = a \oplus_{s_2,s_1} b = b \oplus_{s_1,s_2} a,$$

giving the kind of twisting relation $u^*(a \oplus_{t_1,t_2} b)u = b \oplus_{s_1,s_2} a$ for all $a, b \in A$.
Applying the above at $(\circ)$ below we therefore obtain for all $a \in A$:

$$\begin{aligned}
&||u^*V(a)u - a|| = ||u^*V(a)u - \Big(WV(a) \oplus_{s_1,s_2} V(a)\Big) + \Big(WV(a) \oplus_{s_1,s_2} V(a)\Big) - a|| \\
\overset{(\circ)}{=}&||u^*V(a)u - u^*\Big(V(a) \oplus_{t_1,t_2} WV(a)\Big)u + \Big(WV(a) \oplus_{s_1,s_2} V(a)\Big) - a|| \\
\leq&||u^*\Big(V(a) - \Big(V(a) \oplus_{t_1,t_2} WV(a)\Big)\Big)u|| + ||\Big(WV(a) \oplus_{s_1,s_2} V(a)\Big) - a|| \\
=&||V(a) - \Big(V(a) \oplus_{t_1,t_2} WV(a)\Big)|| + ||s_1 WV(a) s_1^* + s_2 V(a) s_2^* - a|| \\
=&||V(a) - \Big(V(a) \oplus_{t_1,t_2} WV(a)\Big)|| + ||s_1 a s_1^* + s_2 V(a) s_2^* - a + s_1 WV(a) s_1^* - s_1 a s_1^*|| \\
\leq&\underbrace{||V(a) - \Big(V(a) \oplus_{t_1,t_2} WV(a)\Big)||}_{=:x} + \underbrace{||a \oplus_{s_1,s_2} V(a) - a||}_{=:y} + \underbrace{||WV(a) - a||}_{=:z}
\end{aligned}$$

Fix some $a \in F$. To begin with, $z \leq \kappa$ by the very definition of κ. Concerning y we use (i) and (ii) from Lemma 4.26 to see that

$$y \overset{(i)}{\leq} ||[r_1, V(a)]|| + ||[r_2, V(a)]|| + 2||[a, ss^*]|| \overset{(ii)}{\leq} \frac{\varepsilon}{2} + 2\kappa,$$

Remark 4.23 and Lemma 4.26 (ii) give $(*)$:

$$\begin{aligned}
&||[V(a), ss^*]|| \\
\overset{(ii)}{\leq}\ &\max\{||W(V(a)^*V(a)) - WV(a)^*WV(a)||^{1/2}, \\
&\qquad ||W(V(a)V(a)^*) - WV(a)WV(a)^*||^{1/2}\} \\
\leq\ &\max\{||WV(a^*a) - WV(a)^*WV(a)||^{1/2}, \\
&\qquad ||WV(aa^*) - WV(a)WV(a)^*||^{1/2}\} \leq \kappa.
\end{aligned}$$

Again by virtue of 4.26 we have

$$x \overset{(i)}{\leq} ||[r_1, WV(a)]|| + ||[r_2, WV(a)]|| + 2||[V(a), ss^*]|| \overset{(*)}{\leq} \frac{\varepsilon}{2} + 2\kappa.$$

All together we get:

$$||u^*V(a)u - a|| \leq \frac{\varepsilon}{2} + 2\kappa + \frac{\varepsilon}{2} + 2\kappa + \kappa = 5\kappa + \varepsilon.$$

□

Definition 7.18. *Let $\varphi, \psi\colon A \to B$ be *-homomorphisms between unital C^*-algebras A and B. Then φ and ψ are said to be approximately unitarily equivalent if for every finite subset $F \subseteq A$ and $\varepsilon > 0$ there is a unitary $u \in A$ such that*

$$||u^*\varphi(a)u - \psi(a)|| \leq \varepsilon$$

*for all $a \in F$; that is, φ is the point norm limit of *-conjugations of ψ by unitaries.*

By means of Lemma 7.11, we see that in the case where A is separable, and φ and ψ are approximately unitarily equivalent, one may choose a sequence $(u_n)_n$ of unitaries such that $u_n^*\varphi u_n$ converges to ψ pointwise.

Lemma 7.19 (cf. [KiPhi, Lemma 1.10]; [Rr1, Lemma 6.3.7]). *Let A and B be unital C^*-algebras, B separable and $\mathcal{O}_2$-absorbing; assume that $\varphi, \psi\colon A \to B$ are unital *-homomorphisms such that there are sequences of isometries $(s_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ in B such that*

$$||s_n^*\varphi(a)s_n - \psi(a)|| \to 0, \qquad\qquad ||t_n^*\psi(a)t_n - \varphi(a)|| \to 0.$$

for all $a \in A$. Then φ and ψ are approximately unitarily equivalent.

Proof. Let V_n, W_n be the *-conjugations by s_n, t_n respectively. Let $\varepsilon > 0$ and $F \subseteq A$ finite. Without loss of generality F is a subset of the unit sphere by scaling.
The idea is to use the above Lemma to approximate ψ with a *-conjugation of φ by a unitary, first pointwise, and then on all of F; it then suffices to show that κ (cf. Lemma 7.17) tends to zero given an appropriate choice of isometries.
We show that we may assume $\kappa^{s_k,t_k}_{\varphi(F)\cup\varphi(F^*)} \leq \frac{\varepsilon}{15}$ for all $k \geq n$ for some $n \in \mathbb{N}$. First we prove that for every $\varphi(a) \in \varphi(F)$ there is an $n_a \in \mathbb{N}$ with $\kappa^{s_n,t_n}_{\{a\}} \leq \varepsilon/15$ for all $n \in \mathbb{N}_{\geq n_a}$, afterwards we obtain n as $n := \max\{n_a \mid a \in F \cup F^*\} < \infty$.
1) Choose $n_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq n_1}$:

$$\max\{||V_n(\varphi(a^*a)) - \psi(a^*a)||, ||V_n(\varphi(a)) - \psi(a)||\} \leq \frac{\varepsilon}{3\cdot 15}.$$

Then

$$||V_n(\varphi(a)^*\varphi(a)) - V_n(\varphi(a))^*V_n(\varphi(a))||$$

$$\leq ||V_n\varphi(a^*a) - \psi(a^*a)|| + ||\psi(a)^*\psi(a) - V_n(\varphi(a))^*V_n(\varphi(a))|| \leq \frac{\varepsilon}{3\cdot 15} + 2\frac{\varepsilon}{3\cdot 15} = \varepsilon$$

for all $n \in \mathbb{N}_{\geq n_1}$.
2) Choose $n_2 \in \mathbb{N}_{\geq n_1}$ such that for all $n \in \mathbb{N}_{\geq n_2}$:

$$\begin{aligned}\max\{&||V_n(\varphi(a^*a)) - \psi(a^*a)||, ||W_n(\psi(a^*a)) - \varphi(a^*a)||,\\ &||W_n(\psi(a)) - \varphi(a)||, ||V_n(\varphi(a)) - \psi(a)||\} \leq \frac{\varepsilon}{6\cdot 15}.\end{aligned}$$

It follows:

$$\begin{aligned}
&||W_nV_n(\varphi(a)^*\varphi(a)) - W_nV_n(\varphi(a)^*)W_nV_n(\varphi(a))|| \\
\leq&||W_nV_n(\varphi(a^*a)) - \varphi(a^*a)|| \\
&+ ||\varphi(a^*a) - W_nV_n(\varphi(a)^*)W_nV_n(\varphi(a))|| \\
\leq&||W_nV_n(\varphi(a^*a)) - W_n(\psi(a^*a))|| \\
&+ ||W_n(\psi(a^*a)) - \varphi(a^*a)|| \\
&+ ||\varphi(a)^*\varphi(a) - W_n(\psi(a)^*)W_n(\psi(a))|| \\
&+ ||W_n(\psi(a)^*)W_n(\psi(a)) - W_nV_n(\varphi(a)^*)W_nV_n(\varphi(a))|| \\
\leq&\frac{\varepsilon}{15},
\end{aligned}$$

for all $n \in \mathbb{N}_{\geq n_2}$, and:

$$||W_nV_n(a) - a|| \leq ||W_nV_n(\varphi(a)) - W_n(\psi(a))|| + ||W_n(\psi(a)) - \varphi(a)|| \leq \frac{\varepsilon}{15}.$$

Now setting $n_a := \max\{n_1, n_2\}$ will give the desired approximation; define $n := \max\{n_a \mid a \in F \cup F^*\}$.
Now we may take a unitary $u \in B$ as in Lemma 7.17 such that for all $a \in F$

$$||u^*V_n(\varphi(a))u - \varphi(a)|| \leq 5\kappa^{s_n,t_n}_{\varphi(F)\cup\varphi(F^*)} + \varepsilon/3,$$

and as $n \geq n_a$ for all $a \in F$:

$$\begin{aligned}
||u^*\psi(a)u - \varphi(a)|| &\leq ||u^*\psi(a)u - u^*V_n(\varphi(a))u|| + ||u^*V_n(\varphi(a))u - \varphi(a)|| \\
&\leq 5\kappa^{s_n,t_n}_{\varphi(F)\cup\varphi(F^*)} + \varepsilon/3 + \varepsilon/3 \leq \varepsilon.
\end{aligned}$$

□

One could also prove the above Lemma in a more elegant fashion as follows: Let ω be a free filter, denote by $\pi_\omega : l^\infty(B) \to (B)_\omega$ the quotient map onto the ultrapower of B and set $s := \pi_\omega((s_i)_i)$, $t := \pi_\omega((t_i)_i)$, yielding two isometries in $(B)_\omega$. Then, viewing φ and ψ as maps into $(B)_\omega$, we have $s^*\varphi s = \psi$ and $t^*\psi t = \varphi$, and may now apply Lemma 7.17, even assuming that κ is zero. This gives a sequence of unitaries $(u_n)_n$ in the ultrapower of B such that $u_n^*\varphi u_n$ tends to ψ pointwise. A close inspection of the proof of Lemma 7.21 below shows that in case A is separable, approximate unitary equivalence of φ and ψ as maps into the ultrapower of B is equivalent to approximate unitary equivalence as maps in B.
Viewing the sequence implementing approximate unitary equivalence as an element of the ultrapower, φ and ψ are even unitarily equivalent in $(B)_\omega$. Hence, if A is separable, unitary equivalence and approximate unitary equivalence for maps into some ultrapower are equivalent.
One can deal similarly with the case where A is not separable (in order not to loose any generality), and choose for every finite subset of A a unitary $u \in (B)_\omega$ such that $u^*\varphi u \approx \psi$ on F, lift this unitary to a sequence of unitaries $(u_n)_n$, and choose a unitary u_n in this sequence such that $u_n^*\varphi u_n \approx \psi$ on F.

Theorem 7.20 (cf. [Rr1, Theorem 6.3.8]). *Let A be a unital, separable, exact C^*-algebra.*

(i) *Let B be a simple, separable, unital and nuclear C^*-algebra. Then any two injective, unital *-homomorphisms from A into $B \otimes \mathcal{O}_2$ are approximately unitarily equivalent.*

(ii) *Any two invective, unital *-homomorphisms $A \to \mathcal{O}_2$ are approximately unitarily equivalent.*

Proof. Note that the second statement follows from the first and Theorem 4.22.
To prove the first, let $\varphi, \psi : A \to B \otimes \mathcal{O}_2$ be two unital, injective *-homomorphisms. By [Rr1, Theorem 4.1.10] we have that $B \otimes \mathcal{O}_2$ is simple and purely infinite, further it is clear that the minimal tensor product of nuclear C^*-algebras is nuclear, hence $B \otimes \mathcal{O}_2$ is a Kirchberg algebra. By 4.22 we have that $B \otimes \mathcal{O}_2$ is $\mathcal{O}_2$-absorbing, hence by Corollary 7.14 there are sequences of isometries $(s_n)_n$, $(t_n)_n$ such that

$$||s_n^*\varphi(a)s_n - \psi(a)|| \to 0 \qquad \text{and} \qquad ||t_n^*\psi(a)t_n - \varphi(a)|| \to 0$$

as $n \to \infty$. Therefore we may apply Lemma 7.19 to see that φ and ψ are approximately unitarily equivalent. $\square$

Lemma 7.21. *Let A be a separable, unital, exact C^*-algebra and ω an ultrafilter on $\mathbb{N}$. Let $\varphi : A \to (\mathcal{O}_2)_\omega$ be a unital, injective *-homomorphism with a unital, completely positive lift $\rho : A \to l^\infty(\mathcal{O}_2)$; that is, we have a commutative diagram*

$$\begin{tikzcd} & l^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\ A \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & (\mathcal{O}_2)_\omega \end{tikzcd}$$

Then there is a unital embedding of A into $\mathcal{O}_2$.

A rough outline of the proof is as follows: First, we use that the component functions ρ_k of ρ tend to φ, which is isometric, to control the norms of a subsequence of the ρ_k (which we assume to be the ρ_k themselves). Then we are able to apply Proposition 7.13 to these ρ_k and find a $\sigma_k : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\sigma_k \circ \rho_k \approx \rho_{k+1}$, and this σ_k may in turn be approximated by a *-conjugation with an isometry s_k such that $s_k^*\rho_k s_k \approx \rho_{k+1}$. Viewing all these maps as maps in $(\mathcal{O}_2)_\infty$, we will be able to apply Lemma 7.19 and choose a sequence of unitaries such that $u_k^*\rho_k(a)u_k$ converges sufficiently fast to define a *-homomorphism by $\psi(a) := \lim_{k\to\infty} \mathrm{Ad}(u_1 \cdots u_k)\rho_k(a)$. However, matters are not quite that simple in the end, as we have to take care in order to choose the subsequence in a way that preserves approximate multiplicativity.

Proof. First observe that the projection π_k onto the k-th component of $l^\infty(\mathcal{O}_2)$ is a unital *-homomorphism and therefore completely positive, as a consequence, all the $\rho_k := \pi_k \circ \rho : A \to \mathcal{O}_2$ are unital completely positive maps with $||\rho_k||_{cb} = 1$ and $\rho(a) = (\rho_k(a))_k$ for every $a \in A$. Choose an increasing sequence E_j of finite dimensional operator spaces in A with dense union in A. Set $\varepsilon_j := 1/j$ and choose n_j for every $j \in \mathbb{N}$ as in Proposition 7.13 according to $E = E_j$ and $\varepsilon = \varepsilon_j$. Observe that for all $j \in \mathbb{N}$ the map $M_{n_j}(\varphi)$ is an injective, and therefore isometric *-homomorphism, whence

$$\lim_{k \to_\omega \infty} ||M_{n_j}(\rho_k)(a)|| = ||M_{n_j}(\pi_\omega \circ \rho)(a)|| = ||\varphi(a)|| = 1 \tag{7.9}$$

and

$$\lim_{k \to_\omega \infty} ||\rho_k(ab) - \rho_k(a)\rho_k(b)|| = ||\varphi(ab) - \varphi(a)\varphi(b)|| = 0\,. \tag{7.10}$$

For all $j \in \mathbb{N}$ fix some $\delta_j \in \mathbb{R}$ with $(1 - \frac{\varepsilon_j}{2}) < \delta_j < 1$, and choose, using that the unit sphere $S^1(E)$ of a finite dimensional, normed vector space E is compact, an ε_j-dense subset $F_j \subseteq S^1(E_j)$. Set $\delta'_j := (\delta_j - (1 + \frac{\varepsilon_j}{2})^{-1})$ and choose a δ'_j-dense subset $G_j \subseteq S^1\big(M_{n_j}(E_j)\big)$. Then we may choose, by the definition of convergence along a filter and equations 7.9 and 7.10, for every $j \in \mathbb{N}$ some $X_j \in \omega$ such that for all $k \in X_j$, $a, b \in F_j$ and $c \in G_j$:

$$||\rho_k(ab) - \rho_k(a)\rho_k(b)|| < \varepsilon_j\,. \tag{7.11}$$

$$||M_{n_j}(\rho_k)(c)|| \geq \delta_j\,, \tag{7.12}$$

Equation 7.12 implies for all $k \in X_j$ and an arbitrary $d \in S^1\big(M_{n_j}(E_j)\big)$, choosing $d \in G_j$ with $||c - d|| < \delta'_j$, that

$$||M_{n_j}(\rho_k)(d)|| \geq ||M_{n_j}(\rho_k)(c)|| - ||M_{n_j}(\rho_k)(c-d)|| \overset{(*)}{\geq} \delta_j - \delta'_j = (1 + \frac{\varepsilon_j}{2})^{-1}\,, \tag{7.13}$$

where the inequality at $(*)$ follows from $||\rho||_{cb} = 1$. It follows that for all $k \in X_k$ the linear map $\rho_k|_{E_j}$ is injective (as A embeds as the upper right corner in $M_{n_j}(A)$) and

$$||M_{n_j}(\rho_k)^{-1}(c)|| \leq (1 + \frac{\varepsilon_j}{2})$$

for all $c \in S^1\big(M_{n_j}(E_j)\big)$. Now choose for $j \in \mathbb{N}$ some $k_j \in X_j \cap X_{j+1}$ inductively. Then we obtain a sequence $(\rho_{k_1}, \rho_{k_2}, \ldots)$ such that for all $j \in \mathbb{N}$ we can apply Proposition 7.13 to E_j, ε_j, first taking $\rho = \rho_{k_j}$, and $\sigma = \rho_{k_{j+1}}$, and then $\rho = \rho_{k_{j+1}}$, $\sigma = \rho_{k_j}$. In order to keep the notation simple, we assume without loss of generality, that $\rho_{k_j} = \rho_j$ for all $j \in \mathbb{N}$. We hence have unital completely positive maps $\sigma_j, \tau_j : \mathcal{O}_2 \to \mathcal{O}_2$ making the following diagrams

$$\begin{tikzcd} & E_j \arrow[dl, "\rho_j"'] \arrow[dr, "\rho_{j+1}"] & \\ \mathcal{O}_2 \arrow[rr, "\sigma_j"] & & \mathcal{O}_2 \end{tikzcd} \qquad \begin{tikzcd} & E_j \arrow[dl, "\rho_j"'] \arrow[dr, "\rho_{j+1}"] & \\ \mathcal{O}_2 & & \mathcal{O}_2 \arrow[ll, "\tau_j"'] \end{tikzcd}$$

commute within ε_j. In addition, all the η_j, τ_j are nuclear by [Rr1, 6.1.3], and Proposition 7.12 allows us to choose isometries $s_j, t_j \in A$ for all $j \in \mathbb{N}$ such that

$$||s_j^* \rho_j(a) s_j - \sigma_j(\rho_j(a))|| < \varepsilon_j \qquad \text{and} \qquad ||t_j^* \rho_{j+1}(a) t_j - \tau_j(\rho_{j+1}(a))|| < \varepsilon_j$$

for all $a \in F_j$. For an arbitrary $a \in S^1(E_j)$ we choose $b \in F_j$ such that $||a-b|| < \varepsilon_j$ and deduce

$$\begin{aligned} ||s_j^* \rho_j(a) s_j - \rho_{j+1}(a)|| \le ||s_j^* \rho_j(a) s_j - s_j^* \rho_j(b) s_j|| + ||s_j^* \rho_j(b) s_j - \sigma_j \rho_j(b)|| \\ + ||\sigma_j \rho_j(b) - \rho_{j+1}(b)|| + ||\rho_{j+1}(b) - \rho_{j+1}(a)|| \le 4\varepsilon_j \,. \end{aligned}$$

Hence $s_j^* \rho_j(a) s_j \to \rho_{j+1}(a)$ for $j \to \infty$ for all $a \in \cup_{j \in \mathbb{N}} E_j$, and by continuity the same statement holds for all $a \in A$. Similarly $t_j^* \rho_{j+1}(a) t_j \to \rho_j(a)$ as $j \to \infty$ for all $a \in A$. Set $s := (s_j)_j$, $t := (t_j)_j$, $\rho' := (\rho_{j+1})_j$ and denote the quotient map from $l^\infty(\mathcal{O}_2)$ onto $(\mathcal{O}_2)_\infty$ by π_∞. Hence

$$\pi_\infty(s^* \rho(a) s) = \pi_\infty(\rho'(a)) \qquad \text{and similarly} \qquad \pi_\infty(t^* \rho'(a) t) = \pi_\infty(\rho(a)) \,,$$

further $\pi_\infty \circ \rho$ and $\pi_\infty \circ \rho'$ are multiplicative by equation 7.11, and therefore the conditions of Lemma 7.19 are met. Now choose, combining Lemma 7.11 and Lemma A.8, a sequence $(u^{(k)})_k = \big((u_j^{(k)})_j\big)_k$ of unitaries in $l^\infty(\mathcal{O}_2)$ such that

$$\pi_\infty(u^{(k)})^* (\pi_\infty \circ \rho)\, \pi_\infty(u^{(k)}) \to \pi_\infty \circ \rho'$$

pointwise as $k \to \infty$. We will now construct a sequence $(u_i)_i$ of unitaries and a subsequence $(\rho_{j_i})_i$ of ρ such that $||u_i^* \rho_{j_i}(a) u_i - \rho_{j_{i+1}}(a)|| \le \frac{1}{2^i}$ for all $a \in S^1(E_i)$, and it suffices to do this for a finite subset $F_i = \{a_1, \ldots, a_n\}$ of the unit sphere of E_i (using the same techniques as in the construction of the ρ_j above). To begin with, choose $k \in \mathbb{N}$ such that

$$||\pi_\infty(u^{(k)})^* \big(\pi_\infty(\rho(a))\big) \pi_\infty(u^{(k)}) - \pi_\infty(\rho'(a))|| < \frac{1}{2^i}$$

for all $a \in F_i$. Taking $a^{(m)} = \pi_\omega\big((u^{(k)})^* \rho(a_m) u^{(k)} - \rho'(a_m)\big)$ for all $m = 1, \ldots, n$ in Proposition A.7 (iii) we see that there is a subset $Y_i \in \omega_\infty$ such that

$$||(u_j^{(k)})^* \rho_j(a)\, u_j^{(k)} - \rho_{j+1}(a)|| \le \frac{1}{2^i}$$

for all $j \in Y_i$. Choosing some $j_i \in Y_i$ with $j_i \ge j_{i-1}$ inductively and setting $u_i := u_{j_i}^{(k)}$ we thus obtain a sequence as desired.
Now let $a \in S^1(E_i)$ for some $i \in \mathbb{N}$, $n, m \in \mathbb{N}_{\ge i}$ with $n \ge m$ and calculate

$$\begin{aligned} &|| \operatorname{Ad}(u_1 \cdots u_m) \rho_{j_m}(a) - \operatorname{Ad}(u_1 \cdots u_n) \rho_{j_n}(a)|| = ||\rho_{j_m}(a) - \operatorname{Ad}(u_{m+1} \cdots u_n) \rho_{j_n}(a)|| \\ &\le \sum_{l=m}^{n-1} ||\rho_{j_l}(a) - \operatorname{Ad}(u_{l+1}) \rho_{j_{l+1}}(a)|| \le \sum_{l=m}^{n-1} \frac{1}{2^l} \underset{m,n \to \infty}{\longrightarrow} 0 \,. \end{aligned}$$

This entails existence of $\psi(a) := \lim_{i\to\infty} \mathrm{Ad}(u_1\cdots u_i)\rho_{j_i}(a)$ for all $a \in E_i$ by scaling, and hence for all $a \in A$ by continuity; it remains to show that ψ is a unital embedding, i.e., a unital, injective *-homomorphism. It is clear that ψ is linear and unital, as all the ρ_{j_i} are. But $(\rho_{j_i})_i$ is a subsequence of an approximately multiplicative sequence, hence

$$||\psi(ab) - \psi(a)\psi(b)|| = \lim_{i\to\infty} ||\rho_{j_i}(ab) - \rho_{j_i}(a)\rho_{j_i}(b)|| \leq \lim_{i\to\infty} \varepsilon_j = 0$$

or all a in the unit sphere of some E_j; we further have, recalling equation 7.13

$$||\psi(a)|| = \lim_{i\to\infty} ||\rho_{j_i}(a)|| \geq \lim_{i\to\infty} (1 + \frac{\varepsilon_{j_i}}{2})^{-1} = 1\,,$$

and therefore φ is isometric, because it is norm decreasing as a *-homomorphism. □

Of course, one could again apply Lemma 7.17 directly to the sequences ρ and ρ'; this would mean choosing the subsequence of $(\rho_k)_k$ from the beginning such that the constant κ in Lemma 7.17 tends to zero.
The following embedding theorem for separable, quasidiagonal, exact C^*-algebras will be used in the proof of the exact embedding theorem. In the proof of the lemma, we will have to embed a quotient of a product of matrix algebras in $(\mathcal{O}_2)_\omega$; the natural map from this quotient is not always injective though; this may be circumvented by embedding the quasidiagonal C^*-algebra in a product of matrix algebras factored through ω-zero sequences in the product. Details are given in the appendix.

Lemma 7.22 (cf. [Rr1, Lemma 6.3.10]). *Every quasidiagonal, separable, unital , exact C^*-algebra is a sub-C^*-algebra of $\mathcal{O}_2$ up to isomorphism.*

Proof. Let ω be a free ultrafilter on the natural numbers. To begin with, choose a sequence $(k_j)_j$ of natural numbers, a unital completely positive map $\tilde\beta$ such that $\varphi = \tilde\pi_\omega\,\tilde\beta$ is an injective, unital *-homomorphism as in Lemma A.20. Next observe that every matrix algebra M_{k_j} has a unital embedding into $\mathcal{O}_2$ by Lemma 4.21 and Proposition 4.20, hence the product of the M_{k_j} has an embedding ι in $l^\infty(\mathcal{O}_2)$. We therefore have a commutative diagram

$$\begin{tikzcd}
 & \prod_{j=1}^{\infty} M_{k_j} \arrow[r, "\iota"] \arrow[d, "\tilde\pi_\omega"] & l^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\
A \arrow[ur, "\tilde\beta"] \arrow[r, "\varphi"'] & \prod_{j=1}^{\infty} M_{k_j} \Big/ \bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j} \arrow[r, "\bar\iota"'] & (\mathcal{O}_2)_\omega
\end{tikzcd}$$

where $\bar\iota$ is obtained by factoring $\pi_\omega\,\iota$ over $\tilde\pi_\omega$. As ι is an injective, hence isometric, *-homomorphism, a sequence in the domain of ι converges to zero along ω if and only if its image under ι converges to zero along ω, showing that $\bar{\iota}$ is injective. Now $\iota \circ \tilde{\beta}$ is a completely positive unital map as composition of such and $\bar{\iota} \circ \varphi$ is an injective, unital *-homomorphism, consequently we may invoke Lemma 7.21 to see that A embeds into $\mathcal{O}_2$. □

We will also need the following theorem by Choi and Effros in the proof of the embedding theorem:

Theorem 7.23 (Choi-Effros lifting theorem, cf. [Lin, Theorem 5.4.4]). *Let A be a separable C^*-algebra, B a C^*-algebra and I an ideal of B and denote by $\pi\colon B \to B/I$ the quotient map . Then for every nuclear, contractive completely positive map $\varphi : A \to B/I$ there is a contractive completely positive linear map $\psi : A \to B$ such that $\pi \circ \psi = \varphi$.*

We are now prepared to address the main theorem:

Theorem 7.24 (Kirchberg's exact embedding theorem). *Every separable exact C^*-algebra is a sub C^*-algebra of the Cuntz algebra $\mathcal{O}_2$.*

Proof. We only prove the case where A is nuclear, the general case is proved in [KiPhi, Theorem 2.8].
We have to show that there is an injective *-homomorphism from A into $\mathcal{O}_2$. Let again τ denote the *-automorphism on $C_0(\mathbb{R})$ given by $\tau(f)(t) = f(t+1)$. As A embeds into $C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z}$ by Proposition 6.23, it suffices to show that the latter embeds into $\mathcal{O}_2$. Setting $B := \widetilde{C_0(\mathbb{R}, A)}$, we will construct the following sequence

$$C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z} \xhookrightarrow{(1)} B \rtimes_\tau \mathbb{Z} \xhookrightarrow{(2)} (\mathcal{O}_2)_\omega \otimes C(\mathbb{T}) \xhookrightarrow{(3)} (\mathcal{O}_2)_\omega$$

Ad (1): Use the isomorphism from Proposition 6.13.
Ad (2): We construct a *-homomorphism as needed for an application of 6.20. First of all, $C_0(\mathbb{R}, A)$ is exact as it is isomorphic to the tensor product of the nuclear C^*-algebra $C_0(\mathbb{R})$ and the exact C^*-algebra A, therefore $B \rtimes_\tau \mathbb{Z}$ is exact by [Rr1, Proposition 6.1.10]. Further $C_0(\mathbb{R}, A) \cong C_0\big((0,1), A\big)$, and the latter is a sub-C^*-algebra of $C_0\big((0,1], A\big)$, which is quasidiagonal by Example A.23, hence quasidiagonal itself; it follows that B is quasidiagonal as the unitization of a quasidiagonal C^*-algebra. We may therefore apply Lemma 7.20 and obtain a unital embedding ι of B into $\mathcal{O}_2$; for simplicity, we assume that ι is the inclusion map. Let ω be a free ultrafilter. As any two injective, unital *-homomorphisms from B into $\mathcal{O}_2$ are approximately unitarily equivalent by Theorem 7.11, so are ι and $\iota \circ \tau$. Hence we may find a sequence $(u_n)_n$ of unitaries in $\mathcal{O}_2$ such that $u_n b u_n^* \to \tau(b)$ for all $b \in B$. Setting $u := \pi_\omega((u_n)_n)$ and denoting by ι' the canonical inclusion of $\mathcal{O}_2$ into $(\mathcal{O}_2)_\omega$, we get $u\iota'(b)u^* = \iota'(\tau(b))$ for all $b \in B$, that is, as maps in $(\mathcal{O}_2)_\omega$, τ and the inclusion are unitarily equivalent. Thence we may apply Lemma 6.20 to the canonical inclusion $i'|_B$ to obtain the embedding (2).
Ad (3) By Proposition 2.12, $C(\mathbb{T})$ embed into $\mathcal{O}_2$. Combining Lemma A.10 and Theorem 4.22, we see that $(\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$ embeds into $(\mathcal{O}_2)_\omega$.

Now, if A is nuclear, then by [Rr1, Proposition 2.1.2] we have that: $C_0(\mathbb{R})$ is nuclear, hence so is $C_0(\mathbb{R}) \otimes A$ and also B, being an extension of $C_0(\mathbb{R}) \otimes A$ by $\mathbb{C}$, and as $\mathbb{Z}$ is abelian, hence amenable, $B \rtimes_\tau \mathbb{Z}$ is also nuclear. Thus we can use the Choi-Effros lifting theorem (Theorem 7.23) to lift the embedding $\bar{\iota}$ obtained by combining all the embeddings in the diagram above to one unital, completely positive map into $l^\infty(\mathcal{O}_2)$. Therefore, invoking Lemma 7.21 gives an embedding of $B \rtimes_\tau \mathbb{Z}$ in $\mathcal{O}_2$, as desired. $\square$

Remark 7.25. In case where A is unital, the image of A under the embedding constructed above contains a projection p, namely the image of 1_A. As $\mathcal{O}_2$ has trivial K-theory (cf. [Rr1, Equation 4.2.6] or [RrLL, Exercise 4.5]), one may apply [Rr1, Proposition 4.1.4] (or [RrLL, Exercise 4.9]) by simplicity of $\mathcal{O}_2$ to see that p is equivalent to $1_{\mathcal{O}_2}$. Hence there is a partial isometry $v \in \mathcal{O}_2$ such that $vv^* = p$ and $v^*v = 1_{\mathcal{O}_2}$. As the map $a \mapsto v^*av$ from $\mathcal{O}_2$ onto $p\mathcal{O}_2p$ is a unital *-homomorphism, we see that the embedding may be chosen to be unital.

$$
\begin{tikzcd}
 & & & & A \arrow[rrrr, "\text{composition}"] \arrow[d, "a\mapsto p\otimes a"'] & & & & \mathcal{O}_2 \\
 & & & & (C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A \arrow[d, "fu\otimes a\mapsto (f\otimes a)(u\otimes 1)"'] & & & & \\
 & & & & C_0(\mathbb{R}) \otimes A \rtimes_{\tau\otimes \mathrm{id}} \mathbb{Z} \arrow[d, "f\otimes a\mapsto fa"'] & & & & \\
 & & & & C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z} \arrow[d] & & & & \\
\widetilde{C_0(\mathbb{R}, A)} \arrow[rrrr] \arrow[dr, "\beta"] \arrow[drr, "\iota\beta"] \arrow[ddr, "\varphi"'] \arrow[dddrrr, bend left, "\psi"] \arrow[dddrrr, bend right, "\psi\,\tau"'] & & & & \widetilde{C_0(\mathbb{R}, A)} \rtimes_\tau \mathbb{Z} \arrow[uuuurrrr] \arrow[rrrr, "\text{Choi-Effros}"] \arrow[dddd, "\overline{\delta\psi}"] \arrow[ddddr, "\overline{\psi\delta}\otimes 1"] \arrow[ddddrrrr, "\text{composition}"] & & & & l^\infty(\mathcal{O}_2) \arrow[dddd, "\pi_\omega"] \\
 & \prod M_{k_j} \arrow[r, "\iota"] \arrow[d, "\pi_\omega"] & l^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] & & & & & & \\
 & \prod M_{k_j} / \bigoplus^\omega M_{k_j} \arrow[r, "\bar{\iota}"] & (\mathcal{O}_2)_\omega & & & & & & \\
 & & & \mathcal{O}_2 \arrow[dr, "\delta"] & & & & & \\
 & & & & (\mathcal{O}_2)_\omega \arrow[r, "s\mapsto s\otimes 1"'] & (\mathcal{O}_2)_\omega \otimes C(\mathbb{T}) \arrow[r, "\mathrm{id}\otimes\chi"'] & (\mathcal{O}_2)_\omega \otimes \mathcal{O}_2 \arrow[r] & (\mathcal{O}_2 \otimes \mathcal{O}_2)_\omega \arrow[r, "\sim"'] & (\mathcal{O}_2)_\omega
\end{tikzcd}
$$

A Appendix

A.1 Ultrapower algebras

We quickly recall the definitions of filters and ultrapower algebras:
A filter in a set S is a set ω of subsets of S which does not contain the empty set, is closed under finite intersections and has the property that for any $G \subseteq S$ with $F \subseteq G \subseteq S$ for some $F \in \omega$ we have $G \in \omega$. By Zorn's Lemma every filter on a set S is contained in a maximal filter (with respect to inclusion), and such a maximal filter is called an ultrafilter on S. Further ω is called free if $\bigcap\limits_{F \in \omega} F = \emptyset$.
We have the following standard characterization for ultrafilters:

Lemma A.1. *A filter ω on a set S is an ultrafilter if and only if for every subset X of S either X or X^c is an element of ω.*

We collect some properties of filters on $\mathbb{N}$ for further reference in the next proposition:

Proposition A.2. *Let ω be a filter on $\mathbb{N}$. If ω is free, then:*

(i) *For every $k \in \mathbb{N}$ we have $\mathbb{N}_{\geq k} \in \omega$,*

(ii) *All $X \in \omega$ are infinite.*

Proof. If $k \in \mathbb{N}$, then for every $i \in \mathbb{N}_{<k}$ there is, by freeness, some $X_i \in \omega$ not containing i and therefore the intersection of all these X_i is a subset of $\mathbb{N}_{\geq k}$, hence (i) follows.
If $X \in \omega$ now were finite, then we could choose $X' \in \omega$ with $X' \subseteq \mathbb{N}_{\geq \sup X}$ and deduce $\emptyset = X \cap X' \in \omega$, which contradicts the defining properties of a filter. $\square$

Example A.3. Consider the filter ω_∞ of all cofinite subsets of $\mathbb{N}$, that is, of all subsets of $\mathbb{N}$ with finite complement. This is a free filter, by Lemma A.1, which it is not an ultrafilter (take even and odd natural numbers). Note though that every free filter contains ω_∞ by Proposition A.2 (i), hence the cofinite filter is a minimal element in the set of free filters on $\mathbb{N}$. Even more is true, it is a lower bound.
The filters ω_n, $n \in \mathbb{N}$, of subsets of $\mathbb{N}$ containing n, are obviously not free; a fact that might also be deduced from Proposition A.2. ω_n is called the neighborhood filter of $\mathbb{N}$.

Definition A.4. *Let ω be a filter on $\mathbb{N}$ and $(s_n)_n$ a sequence in a topological space S. Then $(s_n)_n$ is said to converge to $s \in S$ along ω if for every neighborhood U of s in S there is some $X \in \omega$ such that $s_n \in U$ for all $n \in X$. If $(s_n)_n$ has a unique*

limit $s \in S$, then we write $\lim_\omega v_n = v$ or $\lim_{n \to_\omega \infty} v_n = v$ if we want to emphasize the parameter in question.
A cluster point along ω of $(s_n)_n$ is an element $s \in S$ such that for every neighborhood U of s in S and $X \in \omega$ there is an $n \in X$ with $s_n \in U$.

Proposition A.5. *Let ω be a filter on $\mathbb{N}$.*

(i) *Every sequence in a compact topological space K has a cluster point along ω.*

(ii) *If ω is an ultrafilter, then any sequence with a cluster point converges along ω.*

(iii) *Every sequence in a compact topological space K converges along any ultrafilter.*

Proof. Let K be a compact topological space and $(s_n)_n$ a sequence in K. Then the sets $K_X := \overline{\{s_n \mid n \in X\}}$ contain a common point $s \in S$ by the finite intersection property (observe that $\emptyset \neq K_{X \cap Y} \subseteq K_X \cap K_Y$). This point is then a clusterpoint by definition.
Now assume that $(s_n)_n$ is a sequence in an arbitrary topological space S which has a clusterpoint $s \in S$ and that ω is an ultrafilter. Then for any neighborhood U of s in S the set $\{n \mid s_n \in U\}$ has to be in ω, as otherwise its complement would be in ω by A.1, which contradicts the definition of a clusterpoint. Finally (iii) follows from (i) and (ii). $\square$

Hence convergence of a sequence along ω_∞ is just the usual notion of convergence.
We further define $l^\infty(A)$ to be the $\mathbb{N}$-fold product over A (i.e., the set of bounded sequences over A with ∞-norm), and denote for every filter ω on $\mathbb{N}$ by $c_0^\omega(A)$ the set of sequences $(a_n)_n$ over A tending to zero along ω. We show that $c_0^\omega(A)$ is a closed, two sided ideal in $l^\infty(A)$: As the product of a bounded sequence with a sequence converging to zero along a filter ω again converges to zero along ω and as $c_0^\omega(A)$ is closed under addition and scalar multiplication by continuity, it only remains to show that for a sequence $\big((x_n^{(i)})_n\big)_i$ in $c_0^\omega(A)$ which converges against $(x_n)_n \in l^\infty(A)$ with respect to the supremum norm, the limit is again in $c_0^\omega(A)$. To do so, choose some $i \in \mathbb{N}$ with $||(x_n^{(i)})_n - (x_n)_n||_\infty \leq \varepsilon$ and $X_\varepsilon \in \omega$ with $||x_n^{(i)}|| \leq \varepsilon$. Then for all $n \in X_\varepsilon$ we have

$$||x_n|| \leq ||x_n - x_n^{(i)}|| + ||x_n^{(i)}|| \leq ||(x_n)_n - (x_n^{(i)})_n||_\infty + ||x_n^{(i)}|| \leq 2\varepsilon\,,$$

and therefore we see that $c_0^\omega(A)$ is a closed two sided ideal in $l^\infty(A)$. On behalf of this fact we go on to define the ultrapower of A with respect to ω as

$$(A)_\omega := l^\infty(A)/c_0^\omega(A)$$

and the following maps:

$$\begin{aligned}
\delta_\infty^A &\colon A \to l^\infty(A),\ a \mapsto (a)_n\,,\\
\pi_\omega^A &\colon l^\infty(A) \to (A)_\omega,\ (a_n)_n \mapsto (a_n)_n + c_0^\omega(A)\,,\\
i_\omega^A &\colon A \to A_\omega,\ a \mapsto \pi_\omega^A \circ \delta_\infty^A(a)\,,
\end{aligned}$$

and write π_ω, i_ω if there is no ambiguity.
We further define, for every filter ω on $\mathbb{N}$ and sequence $(r_n)_n$ of real numbers, the limes superior of $(r_n)_n$ along ω as

$$\limsup_\omega r_n := \inf_{X\in\omega} \sup_{n\in X} r_n \,.$$

Remark A.6. We will use the following argument in the next proof: Assume that A is a *-algebra admitting a complete C^*-norm $||\cdot||$. Then $||\cdot||$ is even the unique pre-C^*-norm on A. For if $|||\cdot|||$ is a pre-C^*-norm on A, then the inclusion ι of A into the completion B of A with respect to $|||\cdot|||$ is isometric, and therefore A is complete with respect to $|||\cdot|||$ and the two norms coincide.

Proposition A.7. *We have the following properties for sequences along filters:*

(i) *Let $(r_n)_n$ be a sequence of real numbers converging to $r\in\mathbb{R}$ along some filter ω on $\mathbb{N}$. Then $\limsup_\omega r_n = r$.*

(ii) *Let ω be a filter on $\mathbb{N}$ and $a=(a_n)_n\in l^\infty(A)$. Then*

$$||\pi_\omega(a)|| = \limsup_\omega ||a_n||$$

and if, in addition, ω is a free ultrafilter, then

$$||\pi_\omega(a)|| = \lim_\omega ||a_n|| \,.$$

(iii) *Let $\varepsilon > 0$ and $a^{(1)} = (a_j^{(1)})_j, \dots, a^{(n)} = (a_j^{(n)})_j$ be elements of $l^\infty(A)$ with $||\pi_\omega(a_j^{(m)})_j)|| < \varepsilon$ for all $m=1,\dots,n$. Then there is $X\in\omega$ with $||a_j^{(m)}|| < \varepsilon$ for all $j\in X$ and $m=1,\dots,n$.*

Proof. Ad (i): For every $i\in\mathbb{N}$ choose some $X_i\in\omega$ with $|r_n - r| < \frac{1}{i}$ for all $n\in X_i$; we may assume that the sequence $(X_i)_i$ is decreasing by intersecting. It is easy to see that

$$r = \inf_{i\in\mathbb{N}} \sup_{n\in X_i} r_n \tag{A.1}$$

and certainly

$$r = \inf_{i\in\mathbb{N}} \sup_{n\in X_i} r_n \geq \inf_{X\in\omega} \sup_{n\in X} r_n = \limsup_\omega r_n$$

holds, as the set we take the infimum of is smaller on the left hand side. To see that the opposite inequality, and hence equality, holds, observe that for any $\varepsilon > 0$ we have some $X\in\omega$ such that $\sup_{n\in X} r_n < \limsup_\omega r_n + \varepsilon$. Hence setting all $X_i' := X\cap X_i$ and using equation A.1 again we obtain $r\leq \limsup_\omega r_n$.
Ad (ii): In the following we denote by $\bar{a}$ the class $\pi_\omega(a)$ in $(A)_\omega$ of a bounded sequence over A. We show that the map

$$|||\cdot||| : (A)_\omega \to \mathbb{R}^+,\ \pi_\omega((a_n)_n) \mapsto \limsup_\omega ||a_n||$$

is a C^*-norm on $(A)_\omega$ and therefore has to be equal to the quotient norm on $(A)_\omega$. It suffices by the above remark to show that this is a pre-C^*-norm. Let $a = (a_n)_n, b = (b_n)_n \in l^\infty(A)$ and $\lambda \in \mathbb{C}$; then we have

$$\begin{aligned}|||\bar{a} + \bar{b}||| =& |||\overline{(a+b)}||| = \limsup_\omega ||a_n + b_n|| = \inf_{X\in\omega} \sup_{n\in X} ||a_n + b_n|| \\ &\leq \inf_{X\in\omega} \sup_{n\in X} ||a_n|| + ||b_n|| = \limsup_\omega ||a_n|| + \limsup_\omega ||b_n|| = |||\bar{a}||| + |||\bar{b}|||\,,\end{aligned}$$

submultiplicativity follows similarly; further

$$|||\overline{\lambda a}||| = \inf_{X\in\omega} \sup_{n\in X} |||\lambda a_n||| = |\lambda|\ |||\bar{a}|||$$

and

$$|||\bar{a}^*||| = \limsup_\omega ||a_n^*|| = |||\bar{a}|||\,,$$

$|||\bar{a}^*\bar{a}||| = |||\bar{a}|||^2$ follows similarly.
If ω is a free ultrafilter, then we get for every $(a_n)_n \in l^\infty(A)$ by (i)

$$|||\overline{(a_n)_n}||| = \limsup_\omega ||a_n|| = \lim_\omega ||a_n||\,.$$

Ad (iii): If we have

$$\inf_{X\in\omega} \sup_{j\in X} ||a_j^{(m)}|| = \limsup_\omega ||a_j^{(m)}|| = ||\pi_\omega\big((a_j^{(m)})_j\big)|| < \varepsilon$$

for all $m = 1, \ldots, n$, then we may choose $X_1, \ldots, X_n \in \omega$ with $\sup_{j\in X_m} ||a_j^{(m)}|| < \varepsilon$ for all $m = 1, \ldots, n$. Hence for all $j \in X_1 \cap \ldots \cap X_n$ we have $||a_j^{(m)}|| < \varepsilon$ for all $m = 1, \ldots, n$. □

Lemma A.8. *Let ω be a filter on $\mathbb{N}$. Every unitary in the ultrapower $(A)_\omega$ of a unital C^*-algebra lifts to a unitary in $l^\infty(A)$.*

Proof. Let $\pi_\omega\big((a_n)_n\big)$ be a unitary in $(A)_\omega$, that is

$$\lim_\omega ||a_n^* a_n - 1|| = \lim_\omega ||a_n a_n^* - 1|| = 0\,,$$

and choose $X \in \omega$ such that $||a_n^* a_n - 1||, ||a_n a_n^* - 1|| < 1$ for all $n \in X$. Then for all $n \in X$ the elements $a_n^* a_n$ and $a_n a_n^*$ are invertible (see the paragraph above 5.3), and hence so is a_n, as it is left and right invertible. For every $n \in X$, define a unitary by $u_n = a_n|a_n|^{-1})$, set $u_n := 1$ for all $n \in X^c$ and define a unitary in $l^\infty(A)$ by $u := (u_n)_n$. We have to show that $\lim_\omega ||a_n - u_n|| = 0$, whereby $\pi_\omega\big((a_n)_n\big) = \pi_\omega\big((u_n)_n\big)$. Let $\varepsilon > 0$, choose $X_\varepsilon \in \omega$ with $||a_n^* a_n - 1||, ||a_n a_n^* - 1|| < \varepsilon$ for all $n \in X_\varepsilon$, and set $X'_\varepsilon := X_\varepsilon \cap X$. Then, for every $n \in X'_\varepsilon$ we have $||a_n - u_n|| < \varepsilon$ by Lemma 2.9 (applied with $p = 1$), and the claim follows. □

Remark A.9. Observe that for a sequence converging along a filter ω we do not automatically have convergence of every subsequence, not even if ω is a free ultrafilter. For example, take the sequence $(x_n)_n$ with $x_n = 0$ for n even and $x_n = n$ for n odd. Let e_n denote the set of even natural numbers above n. Then this generates a filter ω_e, which is free as for any $n \in \mathbb{N}$ we have $n \notin e_{n+1}$, and by Zorn's Lemma we may choose an ultrafilter ω_E containing ω_e, which clearly remains free. Now obviously $(x_n)_n$ converges along ω_e, and hence along the ultrafilter containing it. On the other hand we have the subsequence $(x_{2n-1})_n = (1, 3, 5, \ldots)$, and for every $X \in \omega_E$ we know that X is infinite by freeness, hence $(x_{2n-1})_n$ can not converge along ω_E.
Further, the above example shows that a sequence which is convergent along an arbitrary filter does not have to converge along in the ordinary sense, that is, along the filter of cofinite sets ω_∞ defined in Example A.3.

Lemma A.10. *Let ω be a free ultrafilter. For every unital simple purely infinite C^*-algebra A and simple unital nuclear C^*-algebra B there is an embedding*

$$\iota\colon (A)_\omega \otimes B \to (A \otimes B)_\omega \,.$$

Proof. $(A)_\omega$ is simple and purely infinite by [Rr1, Proposition 6.2.6], therefore the minimal tensor product $(A)_\omega \otimes B$ is simple, and coincides with the maximal tensor product by nuclearity. It therefore suffices to show existence of pointwise commuting *-homomorphisms from $(A)_\omega \times B$ into $(A \otimes B)_\omega$, and then to apply universality of the maximal tensor product; these are given by

$$\begin{aligned} &\varphi\colon (A)_\omega \to (A \otimes B)_\omega \,, \pi_\omega^A\big((a_i)_i\big) \mapsto \pi_\omega^{A\otimes B}((a_i \otimes 1)_i) \,, \\ &\psi\colon B \to (A \otimes B)_\omega, \; b \mapsto \pi_\omega^{A\otimes B}\big((1 \otimes b)_n\big) \,. \end{aligned}$$

By simplicity, the induced map $\varphi \otimes \psi$ on the tensor product is an embedding. $\square$

A.2 Quasidiagonal C^*-algebras

In this section we collect the material on quasidiagonal C^*-algebras needed in the proof of the embedding theorem in section 7.3.
Recall that a net $(a_\lambda)_{\lambda\in\Lambda}$ of bounded operators on some Hilbert space $\mathcal{H}$ converges in the strong operator topology (s.o.t.) to an operator a if and only if for all $\xi \in \mathcal{H}$ we have $a(\xi) = \lim_\lambda a_\lambda(\xi)$. In this section, all limits taken are strong limits, except where noted otherwise. Recall further that a bounded operator $a \in \mathcal{B}(\mathcal{H})$ is positive if and only if the sesquilinear form $\xi \mapsto \langle a(\xi)|\xi\rangle$ is positive, implying that the strong limit of positive operators is positive.
Let $(\mathcal{H}_j)_j$ be a sequence of Hilbert spaces. For every bounded sequence $(a_j)_j$ of bounded operators, where $a_j \in \mathcal{B}(\mathcal{H}_j)$ for every $j \in \mathbb{N}$, we define an operator with diagonal $(a_j)_j$ by

$$\Delta(a_j)_j \colon \prod_{j=1}^{\infty} \mathcal{H}_j \to \prod_{j=1}^{\infty} \mathcal{H}_j \;, (\xi_j)_j \mapsto (a_j(\xi_j))_j \,.$$

Then

$$\sum_{j=1}^{\infty} ||a_j\xi_j||^2 \leq \sum_{j=1}^{\infty} ||a_j||^2 ||\xi_j||^2 \leq \sup_j ||a_j||^2 \sum_{j=1}^{\infty} ||\xi_j||^2 \,,$$

showing that $\Delta(a_j)_j$ is a well defined bounded operator. Let $(k_j)_j$ be a sequence of natural numbers. We define

$$\bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j} := \{(a_j)_j \mid (a_j)_j \in \prod_{j=1}^{\infty} M_{k_j},\ \lim_\omega a_j = 0\} \,.$$

Definition A.11. *Let A be a separable C^*-algebra. Then A is called quasidiagonal (QD) if there exists a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$ as bounded operators on some Hilbert space $\mathcal{H}$ such that there is an increasing sequence (with respect to $\leq$) $(p_n)_n$ of finite dimensional projections in $\mathcal{B}(\mathcal{H})$ subject to*

(i) *$(p_n)_n$ is approximately central, i.e., $||[\pi(a), p_n]|| \to 0$ as $n \to \infty$ for all $a \in A$*

(ii) *The sequence $(p_n)_n$ converges strongly to the identity $I \in \mathcal{B}(\mathcal{H})$.*

We will call a representation π of A quasidiagonal, if $\pi(A)$ is quasidiagonal.

Remark A.12. Observe that the unitization functor $\tilde{\cdot}$ maps quasidiagonal C^*-algebras to such, as for A quasidiagonal we may set $\tilde{p}_i := (p_i, 1)$ to obtain for all $\tilde{a} = (a, \lambda) \in \tilde{A}$

$$||[\tilde{p}_i, \tilde{a}]|| = ||(p_i a + \lambda p_i + a, \lambda) - (a p_i + \lambda p_i + a, \lambda)|| = ||[p_i, a]|| \to 0 \,.$$

Further it is obvious that every sub-C^*-algebra of a quasidiagonal C^*-algebra is quasidiagonal.

Proposition A.13. *Let A be a separable quasidiagonal C^*-algebra, π a faithful representation on a Hilbert space $\mathcal{H}$ such that there is an increasing, approximately central sequence of finite dimensional projections from the definition of quasidiagonality. Then there is a sequence $(q_j)_j$ of pairwise orthogonal, finite rank projections in $\mathcal{B}(\mathcal{H})$ such that*

(i) $\sum_{j=1}^{\infty} q_j = 1$,

(ii) *for all $a \in A$ we have $\pi(a) - \sum_{j=1}^{\infty} q_j \pi(a) q_j \in \mathcal{K}(\mathcal{H})$,*

(iii) *for all $a \in A$, $\sum_{j=1}^{\infty} ||[\pi(a), q_j]||$ is convergent.*

*In addition, given such a sequence $(q_j)_j$ as above, the map $\beta\colon A \to \mathcal{B}(\mathcal{H})$ defined by $a \mapsto \sum_{j=1}^{\infty} q_j \pi(a) q_j$ is essentially a *-homomorphism, i.e., a *-homomorphism as a map in $\mathcal{Q}(H) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.*

Proof. For simplicity, we assume $A \subseteq \mathcal{B}(\mathcal{H})$. As A is separable, we may choose an increasing sequence E_j of finite dimensional operator spaces in A with dense union in A; further let $(p_i)_i$ be an increasing, approximately central sequence of finite rank projections in $\mathcal{B}(\mathcal{H})$ and set $p_0 := 0$. We choose inductively, using the same standard arguments as for example in Lemma 7.21, a subsequence $(p_{i_j})_j$ of $(p_i)_i$ so that for $q_j := p_{i_j} - p_{i_{j-1}}$ we have $||[q_j, a]|| \leq \frac{1}{2^j}$ for all $a \in S^1(E_j)$, where $S^1(E_j)$ is again the unit sphere of E_j. Then it is clear that the q_j are pairwise orthogonal and that (i) and (iii) holds for this sequence. We show that β is well defined; if $\xi \in \mathcal{H}$ and $a \in A$, then as the q_j are pairwise orthogonal we have

$$||\sum_{j=1}^{\infty} q_j a q_j \xi||^2 = \sum_{j=1}^{\infty} ||q_j a q_j \xi||^2 \leq ||a||^2 \sum_{j=1}^{\infty} ||q_j \xi||^2 = ||a||^2 \, ||\xi||^2$$

implying that the limit in the strong operator topology exists. For all $n \in \mathbb{N}$ and $a \in M_n(A)^+$ we have

$$M_n(\beta)(a) = \lim_{k\to\infty} \sum_{j=1}^{k} (1_n \otimes q_j) a (1_n \otimes q_j) \geq 0$$

as the sum and strong limit of positive elements is positive; this implies that β is completely positive, and therefore continuous with respect to the norm topology. We show that $a - \beta(a)$ is compact for every $a \in A$. To begin with, let $a \in S^1(E_j)$ for some $j \in \mathbb{N}$. Then

$$a - \beta(a) = a \sum_{j=1}^{\infty} q_j - \sum_{j=1}^{\infty} q_j a q_j = \sum_{j=1}^{\infty} a q_j - q_j a q_j = \sum_{j=1}^{\infty} [a, q_j] q_j \,. \tag{A.2}$$

But as

$$|| \sum_{j=m}^{n} [a, q_j] q_j \, || \leq \sum_{j=m}^{n} || \, [a, q_j] \, || \leq \sum_{j=m}^{n} \frac{1}{2^j} \underset{n,m\to\infty}{\longrightarrow} 0 \,, \tag{A.3}$$

we see that the limit in A.2 is actually a norm limit of compact operators, and therefore compact. As noted before, β is continuous with respect to the norm topology, and therefore the result holds for all $a \in A$ by the usual density arguments. Now it remains to show that β is essentially multiplicative: Let $a, b \in A$. Then we have to show $\beta(ab) - \beta(a)\beta(b) \in \mathcal{K}(\mathcal{H})$, but this follows from

$$\beta(ab) - \beta(a)\beta(b) = \beta(ab) - ab + a(b - \beta(b)) + (a - \beta(a))\beta(b)$$

as all terms in this sum are compact operators by (ii). □

Remark A.14. Note that the subsequence $(p_{i_j})_j$ of the sequence $(p_i)_i$ as chosen in the proof above again has the properties stated in the definition of quasidiagonality. Hence we may always assume that the sequence of orthogonal projections defined above is given by $q_i := p_i - p_{i-1}$, where $p_0 := 0$.

Definition A.15. *Let a be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then a is called block diagonal, if there is a sequence $(p_i)_i$ of pairwise orthogonal, finite rank projections in $\mathcal{B}(\mathcal{H})$ such that $[a, p_i] = 0$ for all $i \in \mathbb{N}$ and such that $\sum_{i=1}^{\infty} p_i = 1$ with respect to the strong operator topology.*

Note that an increasing sequence as above is necessarily an approximate unit (see the paragraph above 3.2) for the compact operators. Hence $C^*(a) + \mathcal{K}(\mathcal{H})$ is a quasidiagonal C^*-algebra for every quasidiagonal operator a.

Corollary A.16. *Let A be a C^*-algebra with representation $\pi : A \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then the following properties are equivalent:*

(i) *π is quasidiagonal.*

(ii) *There is a sequence $(q_j)_j$ of pairwise orthogonal, finite rank projections in $\mathcal{B}(\mathcal{H})$ with $\sum_{j=1}^{\infty} q_j = 1$ and $\kappa(a) := \pi(a) - \sum_{j=1}^{\infty} q_j a q_j \in \mathcal{K}(\mathcal{H})$.*

Proof. The forward implication follows from Proposition A.13. Assume (ii) holds. Then setting $p_i := \sum_{j=1}^{i} q_j$ we have an increasing sequence of finite rank projections by Lemma 2.5, which converges strongly to the identity. Set $\beta(a) := \pi(a) - \kappa(a)$. Then

$$[\pi(a), p_i] = [\beta(a), p_i] - [\kappa(a), p_i] = -[\kappa(a), p_i] \to 0$$

by the preceding paragraph. □

Corollary A.17. *An operator $a \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is quasidiagonal if and only if it is the sum of a block diagonal operator with a compact operator.*

Proof. Proposition A.13 proves the one implication, as β decomposes a into its block diagonal and compact part; the other is obvious by the discussion above. □

Definition A.18. *A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ of A on a Hilbert space $\mathcal{H}$ is called essential if $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.*

There are some subtleties concerning representations of quasidiagonal C^*-algebras: The existence of a quasidiagonal representation does not imply, that every representation will be quasidiagonal. We have the following easy lemma though, concerning quasidiagonality and being essential:

Lemma A.19. *Let A be a C^*-algebra with a representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Then the derived diagonal representation*

$$\tilde{\pi}\colon A \to \mathcal{B}(\prod_{j=1}^{\infty} \mathcal{H}),\ a \mapsto \Delta\,\pi(a) = \Delta\big((\pi(a))_{i\in\mathbb{N}}\big)$$

is an essential representation, which is quasidiagonal, faithful, unital if π was quasidiagonal, faithful, unital, respectively.

Proof. The representation is obviously essential, and faithful, if π was. That the representation is quasidiagonal if π was follows easily if, given an approximately central and increasing sequence of projections (p_i) which converges strongly to the identity, we define $\tilde{p}_i := \Delta(p_i, \ldots, p_i, 0\ldots)$ as a new such sequence for $\tilde{\pi}(A)$. Then the $\tilde{p}_i$ are an increasing sequence of finite dimensional projections, and approximately central, as

$$||[\tilde{p}_i, \Delta\pi(a)]|| = ||\Delta[p_i, \pi(a)]|| = ||[p_i, \pi(a)]||\,,$$

and the latter tends to zero. Further, let $\xi \in \prod_{j=1}^{\infty} \mathcal{H}$, then

$$||\tilde{p}_i(\xi) - \mathrm{id}_{\prod\mathcal{H}}(\xi)|| = ||\tilde{p}_i(\xi) - \Delta((\mathrm{id}_{\mathcal{H}})_j)(\xi)||^2 = \sum_{m=1}^{i} ||p_i(\xi_m) - \xi_m||^2 + \sum_{m=i+1}^{\infty} ||\xi_m||^2\,.$$

Let $\varepsilon > 0$. Choose $I \in \mathbb{N}$ such that for all $i \geq I$ we have $\sum_{l=i+1}^{\infty} ||\xi_l|| \leq \varepsilon$, and $J \in \mathbb{N}$ such that $||p_j\xi_i|| \leq \varepsilon/I$ for all $i \leq I$ and $j \geq J$. Then for all $N > \max\{I, J\}$ we get

$$\sum_{k=1}^{N} ||p_N\xi_k - \xi_k|| + \sum_{l=N+1}^{\infty} ||\xi_l|| \leq \sum_{k=1}^{I} \varepsilon/I + \sum_{k=I+1}^{N} ||p_N\xi_k - \xi_k|| + \sum_{l=N+1}^{\infty} ||\xi_l||$$

$$\leq \sum_{k=1}^{I} \varepsilon/I + \sum_{k=I+1}^{\infty} ||\xi_k|| \leq 2\varepsilon\,.$$

Hence the sequence $(\tilde{p}_i)_i$ converges strongly to the identity. □

Lemma A.20. *Let A be a unital quasidiagonal C^*-algebra and ω a free ultrafilter on $\mathbb{N}$. Then there exists a sequence $(k_j)_j$ of natural numbers and a unital, injective *-homomorphism*

$$\varphi\colon A \to \prod_{j=1}^{\infty} M_{k_j} \Big/ \bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j}$$

with a unital, completely positive lift $\tilde{\beta}\colon A \to \Pi_{j=1}^{\infty} M_{k_j}$; that is, the following diagram, where $\tilde{\pi}_\omega$ denotes the quotient map

$$\begin{tikzcd} & \prod_{j=1}^{\infty} M_{k_j} \arrow[d, "\tilde{\pi}_\omega"] \\ A \arrow[ur, "\tilde{\beta}"] \arrow[r, "\varphi"'] & \prod_{j=1}^{\infty} M_{k_j} \Big/ \bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j} \end{tikzcd}$$

commutes.

Proof. Choose a quasidiagonal, faithful representation π of A on some Hilbert space $\mathcal{H}$ and a sequence $(p_i)_i$ as in the definition of quasidiagonality. We obtain a new quasidiagonal faithful representation $\Delta\pi$ from A to $\mathbb{B} := \mathcal{B}(\prod_{j=1}^{\infty} \mathcal{H})$ as in Lemma A.19 and a new sequence of projections $\tilde{p}_i$ as in the according proof. Define the map $\beta : \mathbb{B} \to \beta(\mathbb{B})$ with respect to $\tilde{q}_i := \tilde{p}_i - \tilde{p}_{i-1}$ as in Proposition A.13 by $a \mapsto \sum \tilde{q}_i a \tilde{q}_i$ (convergence follows as in A.13). Further set $k_i := \dim \tilde{q}_i$ and denote by Δ the isomorphism from $\prod_{j=1}^{\infty} M_{k_j}$ to $\beta(\mathbb{B})$, by $\tilde{\pi}_\omega$ the quotient map

$$\tilde{\pi}_\omega : \prod_{j=1}^{\infty} M_{k_j} \to \prod_{j=1}^{\infty} M_{k_j} \Big/ \bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j} \,.$$

Now set $\tilde{\beta} := \Delta^{-1}\beta\Delta\pi$. We show injectivity of $\tilde{\pi}_\omega\tilde{\beta}$. Assume that $\tilde{\pi}_\omega\tilde{\beta}(a) = 0$ for some $0 \neq a \in A$. Then $\tilde{\beta}(a)$ is a sequence of matrices converging to zero along ω in norm, hence $\Delta\tilde{\beta}(a) = \beta\Delta\pi(a)$ is a blockdiagonal operator whose diagonal sequence has this property. Denoting by $\mathbb{K}$ the compact operators in $\mathbb{B}$, we may use Corollary A.16 to choose $K \in \mathbb{K}$ such that $\Delta\pi(a) = \beta\Delta\pi(a) + K$. The sequence $(\tilde{q}_i)_i$ is an approximate unit for $\mathbb{K}$, hence we see that

$$\lim_{i \to_\omega \infty} ||(\tilde{q}_i\beta\Delta\pi(a)\tilde{q}_i + \tilde{q}_i K \tilde{q}_i)|| = \lim_{i \to_\omega \infty} ||\tilde{\beta}(a)(i)|| = 0 \,.$$

We also have by Proposition 2.2 and the definition of $\tilde{p}_i$

$$||\tilde{q}_i\Delta\pi(a)\tilde{q}_i|| = ||(\tilde{p}_i - \tilde{p}_{i-1})\Delta\pi(a)(\tilde{p}_i - \tilde{p}_{i-1})|| \geq ||p_i\pi(a)p_i|| \,,$$

and therefore, again by Proposition 2.2

$$0 = \lim_{i \to_\omega \infty} ||\tilde{q}_i(\beta\Delta\pi(a) - K)\tilde{q}_i|| \geq \lim_{i \to_\omega \infty} ||p_i\pi(a)p_i|| \geq 0 \,.$$

But as $(\tilde{p}_i)_i$ converges in the strong operator topology to the identity and multiplikation is strongly continuous on the unit ball, $p_i\pi(a)p_i$ converges strongly to $\pi(a)$, which is a contradiction.
It remains to show that $\varphi := \tilde{\pi}_\omega\tilde{\beta}$ is a *-homomorphism, but this follows easily, as β is essentially multiplicative, as shown in Proposition A.13, and Δ is a *-isomorphism $\square$

One could also rephrase this proof more conceptually by introducing some terminology: Call an operator which is the sum of a compact operator and an operator of the form $\Delta(a_j)_j$ for some $(a_j)_j \in \bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j}$ an ω-compact operator, and the representation $\Delta\pi$ ω-essential, if its image does not contain any ω-compact operators (hence, the ω-compact operators are exactly the compact perturbations of images of β whith ω-zero convergent diagonal sequence). Then we have shown above, that the representation $\Delta\pi$ is even ω-essential. Denote these ω-compact operators by $\mathbb{K}^{\omega}$ and by $\pi_{\omega} \colon \mathbb{B} \to \mathbb{B}/\mathbb{K}^{\omega}$ the quotient map. Observe also that $\pi_{\omega}\, \iota_{\beta}\, \Delta$ factors as $\bar{\Delta}\, \tilde{\pi}_{\omega}$, where $\bar{\Delta}$ is injective. Hence the proof may be given as a diagram chase in the following diagram, which commutes on the outside only though, as the upper triangle does not commute (but it does ω-essentially commute, that is, up to an ω-compact operator)

$$\begin{tikzcd}
A \arrow[r, "\Delta\pi"] \arrow[dd, equal] & \Delta\pi(A) \arrow[rr, hook, "\iota_{\Delta\pi}"] \arrow[dr, "\beta|_{\Delta\pi(A)}"'] & & \mathbb{B} \arrow[r, "\pi_{\omega}"] & \mathbb{B}/\mathbb{K}^{\omega} \\
& & \beta(\mathbb{B}) \arrow[ur, hook, "\iota_{\beta}"'] & & \\
A \arrow[rr, "\tilde{\beta}"'] & & \prod_{j=1}^{\infty} M_{k_j} \arrow[u, "\Delta"'] \arrow[rr, "\tilde{\pi}_{\omega}"'] & & \prod_{j=1}^{\infty} M_{k_j} \Big/ \bigoplus_{j=1}^{\infty}{}^{\omega} M_{k_j} \arrow[uu, "\bar{\Delta}"']
\end{tikzcd}$$

Note that the key to this proof was, in fact, the mere existence of an ω-essential representation.

Definition A.21. *Let $\varphi, \psi \colon A \to B$ be *-homomorphisms between C^*-algebras. Then φ and ψ are called homotopic if there is a path of *-homomorphisms $\varphi_t \colon A \to B$ such that for every $a \in A$ the map $t \mapsto \varphi_t(a)$ is continuous.*

We cite the following theorem by Voiculescu, which shows that quasidiagonality is homotopy invariant:

Theorem A.22 (cf. [Vo, Theorem 5]). *Let A and B be C^*-algebras such that there are *-homomorphisms $\varphi \colon A \to B$ and $\psi \colon B \to A$ with $\psi\,\varphi$ homotopic to id_A and B quasidiagonal. Then A is quasidiagonal.*

Example A.23. For every C^*-algebra A the cone $C_0\big((0,1], A\big)$ is quasidiagonal. This follows, as $C_0\big((0,1]\big)$ is zero homotopic, hence so is $C_0\big((0,1]\big) \otimes A \cong C_0\big((0,1], A\big)$, as the spatial norm is a cross norm (i.e., multiplicative with respect to $\otimes$).

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